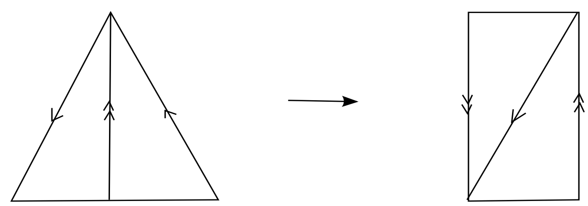
HW 7 SOLUTIONS, MA525

1. Hatcher 2.1, Problem 1



Cut along the edge marked with double arrows, and then flip vertically the triangle on the right, move it to the other side and glue. This shows it to be a mobius band.

2. Hatcher 2.1, Problem 12

Letting $F = 0$ be the zero-homomorphisms, we see that $f_\# - f_\# = 0 = \partial 0 + 0\partial = \partial F + F\partial$. Thus $f \sim f$. If $f \sim g$, then there exists F such that $f_\# - g_\# = \partial F + F\partial$ implying $g_\# - f_\# = -\partial(F) + (-F)\partial = \partial(-F) + (-F)\partial$. Thus $-F$ shows that $g \sim f$. Finally, assume that $f \sim g$ and $g \sim h$. Then there exists homomorphisms F, G such that $f_\# - g_\# = \partial F + F\partial$ and $g_\# - h_\# = \partial G + G\partial$. If we add these equation we obtain $f_\# - h_\# = \partial F + F\partial + \partial G + G\partial$. Since ∂ is a homomorphism this equals $\partial(F + G) + F\partial + G\partial$, and by definition of $F + G$ this equals $\partial(F + G) + (F + G)\partial$. Hence $F + G$ shows that $f \sim h$. Thus we see $\sim$ is reflexive, symmetric and transitive, and hence an equivalence relation.

3. Chapter 2.1, Problem 8

Refer to the picture of X in Hatcher, before the gluing of the *lower* face of T_i to the *upper* face of T_{i+1}. The terms upper and lower make sense in the picture and we shall use them with impunity. We shall call the *central* edge (the axis about which the whole picture can be rotated) as c and it's top vertex as 0 and bottom vertex as $n + 1$. We shall index the vertices in the horizontal plane from 1 to n.

The gluing identifies 0 with $n + 1$ to give a single vertex v. It also identifies vertex i to $i + 1$ which means it identifies all vertices 1 to n to a single vertex w. Thus $C_0 = \langle v, w\rangle \cong \mathbb{Z}^2$.

Moving onto the edges, there is the "central" vertical edge c. There are n edges $[i, 0]$ on top and n edges $[i, n + 1]$ on bottom. The gluing process identifies the bottom edge $[i, n + 1]$ with the top edge $[i + 1, 0]$. Each such pair gives an edge a_i in X. Finally, the horizontal edges running along the "rim" all get identified to a single edge b. Thus $C_1 = \langle b, c, a_i, i = 1, \dots n\rangle \cong \mathbb{Z}^{n+2}$.

From the figure in Hatcher, the left face of each T_i is identified with the right face of T_{i+1} giving n vertical faces S_i, where $\partial(S_i) = a_i + c - a_{i+1}$. The bottom face of T_i is paired with the top face of T_{i+1} by the gluing process, thus giving n horizontal faces R_i, where $\partial(R_i) = a_i - a_{i+1} + b$. Hence $C_2 = \langle S_i, R_i, i = 1, \dots n\rangle \cong \mathbb{Z}^{2n}$.

As neither process identifies tetrahedra, there are still the n 3-simplices T_i, where $\partial(T_i) = S_i - S_{i+1} - R_i + R_{i+1}$. Hence $C_3 = \langle T_i, i = 1, \dots n\rangle$. There are no 4-simplices, so $C_4 = 0$.

$H_0(X) = ker\partial_0/im\partial_1$**:** First observe that $\partial_1(b) = v - v = 0$, $\partial_1(c) = w - w = 0$, and $\partial_1(a_i) = w - v$. Also, since $\partial_0 = 0$, it follows that $ker\partial_0 = C_0 = \langle v, w\rangle$. Hence

$$H_0(X) = \langle v, w \mid w - v = 0\rangle = \langle w - v, w \mid w - v = 0\rangle \cong \mathbb{Z}.$$

$H_1(X) = ker\partial_1/im\partial_2$**:** First note as seen above that $L_1 = \{\{a_{i+1} - a_i\}_{i=1}^{n-1}, b, c\}$ are all in $ker\partial_1$. Since $rk(im\partial_1) = 1$ (as seen above), and $rk(C_1) = n + 2$, and C_1 is finitely generated abelian, it follows by the rank nullity theorem ($rk(C_1) = rk(ker\partial_1) + rk(im\partial_1)$, where rk is the rank, i.e. number of Z summands), that it suffices to show that L_1 is linearly independent to verify that L_1 spans $ker\partial_1$. (We are now viewing all abelian groups as $\mathbb{Z}$-modules, an analog of vector spaces). However, since the a_i, b, c are a basis, it is clear that any linear combination of the elements of L_1 which equals zero must have the coefficiants of c and b being zero. For this same reason, since the term $a_{n-1} - a_n$ is the only term containing an a_n, its coefficiant must also be zero. But now the only remaining term containing a_{n-1} is $a_{n-2} - a_{n-1}$, and by the same reasoning its coefficiant must be zero. Cascading thought the terms in this fashion we see that all coefficients must be zero. Hence L_1 is a basis implying $ker\partial_1 = \langle L_1\rangle$. From our original observations we have

$$im\partial_2 = \langle\{\partial_2(S_i), \partial_2(R_i)\}_{i=1}^{n}\rangle = \langle\{a_i - a_{i+1} + b, a_i - a_{i+1} + c\}_{i=1}^{n}\rangle = \langle\{a_i - a_{i+1} + b\}_{i=1}^{n}, c - b\rangle,$$

where the second equality comes from the fact that we can combine the elements listed on the RHS to get all elements listed on the LHS. Since indices are mod n we have $\sum_{i=1}^{n} a_i - a_{i+1} + b = nb$. Hence $im\partial_2 = \langle\{a_i - a_{i+1} + b\}_{i=1}^{n-1}, c - b, nb\rangle$. Thus we see $H_1(X) = \langle\{a_{i+1} - a_i\}_{i=1}^{n-1}, b, c \mid c - b = 0, nb = 0, a_{i+1} - a_i - b = 0, i \leq n - 1\rangle$, which equals (changing generators, note we can combine new generators to get the old ones) $\langle\{a_{i+1} - a_i - b\}_{i=1}^{n-1}, b, c - b \mid c - b = 0, nb = 0, a_{i+1} - a_i - b = 0\rangle \cong \mathbb{Z}_n$.

$H_2(X) = ker\partial_2/im\partial_3$**:** Note that $\partial_2(-S_i + S_{i+1} + R_i - R_{i+1}) = (-a_i + a_{i+1} - c) + (a_{i+1} - a_{i+2} + c) + (a_i - a_{i+1} + b) + (-a_{i+1} + a_{i+2} - b) = 0$. Hence $L_2 = \{-S_i + S_{i+1} + R_i - R_{i+1}\}_{i=1}^{n-1} \subseteq ker\partial_2$. By the same arguments used for showing L_1 was linearly independent, we conclude that L_2 is a linearly independent set of $n - 1$ elements. From the last paragraph we can conclude from rank-nullity that $rk(im\partial_2) = rk(ker\partial_1) - rk(H_1(X)) = n + 1$, implying from rank-nullity that $rk(ker\partial_2) = rk(C_2) - rk(im\partial_2) = 2n - (n + 1) = n - 1 = |L_2|$. Thus we see that L_2 is a basis for $ker\partial_2$. Observing that $\{\partial_3(T_i)\}_{i=1}^{n} = \{-S_i + S_{i+1} + R_i - R_{i+1}\}_{i=1}^{n} \supseteq L_2$, we conclude that $H_2(X) = 0$.

$H_3(X) = \ker\partial_3/im\partial_4 = \ker\partial_3$**:** Again we have (from previous work) $rk(im\partial_3) = rk(ker\partial_2) - rk(H_2) = (n - 1) - 0 = n - 1$. Hence $rk(ker\partial_3) = rk(C_3) - rk(im\partial_3) = n - (n - 1) = 1$. Thus (since subgroups of free groups are free) we conclude $ker\partial_3 \cong \mathbb{Z}$, and so $H_2(X) \cong \mathbb{Z}$.