HW 6 SOLUTIONS, MA525

Hatcher 1.3 Problem 20

The fundamental group G of the Klein bottle X has the presentation $< a, b \mid abab^{-1} >$. Since $a^3ba^3b^{-1} = a^2(abab^{-1})ba^2b^{-1} = a^2ba^2b^{-1} = a(abab^{-1})bab^{-1} = abab^{-1} = 1$, the map $a \to a^3, b \to b$ induces a homomorphism $\phi : G \to G$. Consider the subgroup $H = \phi(G)$. The covering map $f : X \to X$ associated to H, has degree 3 as shown below

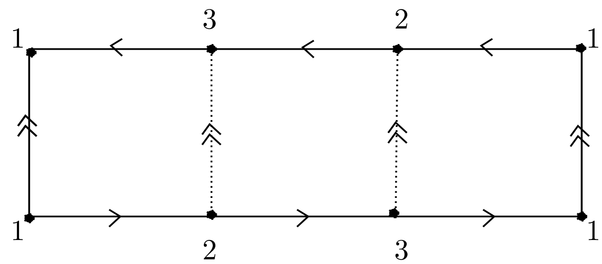

Figure 1. 3-fold cover

where each square gets mapped to the standard Klein bottle. The points marked 1, 2, 3 are the pre-images of the base-point. Consider the lifts of b. Based at 1, b lifts to a loop. Based at 2, it lifts to a path from 2 to 3, and vice versa. This shows that b is in $f_*(\pi(X,1))$ but $b \notin f_*(\pi(X,2))$. Thus, the covering is non-normal. Algebraically, this follows from the fact that $aba^{-1} = a^2b$, which is not in H.

The standard 2-fold covering of the Klein bottle corresponds to the subgroup $\mathbb{Z} \oplus \mathbb{Z}$ generated by a, b^2. Conjugation by b has the action $bab^{-1} = a^{-1}$. So we need to find a rank two subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ that is not invariant under $a \to a^{-1}$. The subgroup generated by a^3 and a^2b^2 works, and represents a non-normal covering by a torus.

Hatcher 1.3 Problem 23

For the action to be a covering space action, it is necessary that it be free. By proper discontinuity, there exists an open set U containing x such that $gU \cap U \neq \varnothing$ for finitely many group elements $g_1 \cdots, g_n$. Because X is Hausdorff, there exist open set $V_0 \subseteq U$ containing x and open sets $V_k \subseteq g_kU$ containing g_kx such that any there are all disjoint (the Hausdorff condition extends to any finite set of distinct points). Now, for $k \geq 1$, let $W = \cap g_k^{-1}(V_k \cap g_kV_0)$. Then W is an open set containing x such that all translates gW are disjoint.

Finally, if the group is finite and the action is free on a Hausdorff, then it is properly discontinuous.