HW 4 SOLUTIONS, MA525

1.2 Problem 7

Let Z denote the quotient space of S^2 obtained by identifying the north pole to the south pole. The space Z is the space marked as X/A in the figure next to Example 0.8 of Hatcher. The cell complex structure is as follows. Start with 1 zero-cell. Attach 1 one-cell by identifying both endpoints of the one-cell to the zero cell (to get S^1). Call the generator of π_1 of the 1-skeleton by a. Then glue 1 two-cell (thought of as a disc with boundary S^1) as shown in the figure below. Here the dots go to the 1-cell. Going around S^1 counter-clockwise,

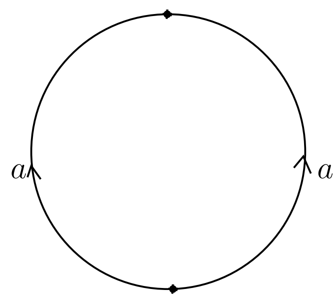

Figure 1. The two-cell

the attaching map sends it to $a^{-1}a = 1$ in π_1 of the 1-skeleton. Hence π_1 of Z is the same as the π_1 of the 1-skeleton i.e. it is $\mathbb{Z}$.

1.2 Problem 9

Part (a): Suppose there is a retraction $r : M'_h \to C$, and let $\imath : S^1 \to M'_h$ denote the inclusion map of C. Consider the induced maps for the composite $r \circ \imath$ at the level of the fundamental groups.

$$\pi_1(S^1) \xrightarrow{\imath_\#} \pi_1(M'_h) \xrightarrow{r_\#} \pi_1(C)$$

Since r is a retraction, the composition $r_\# \circ \imath_\#$ sends the generator of $\pi_1(S^1)$ to the generator of $\pi_1(C)$. Secondly, since $\pi_1(C)$ is abelian the sequence factors through the abelianization G of $\pi_1(M'_h)$ i.e. $r_\# \circ \imath_\#$ can be written as

$$\pi_1(S^1) \xrightarrow{\imath_\#} \pi_1(M'_h) \xrightarrow{q} G \xrightarrow{R_\#} \pi_1(S^1)$$

In $\pi_(M'_h)$, the loop C is the product of commutators $[a_1, b_1][a_2, b_2] \cdots [a_h, b_h]$ (reason: puncturing a closed surface M_h allows deformation retract onto the 1-skeleton). So under the map q, this element is sent to the identity. Hence the composition cannot send the generator of $\pi_1(S^1)$ to $\pi_1(C)$. Contradiction.

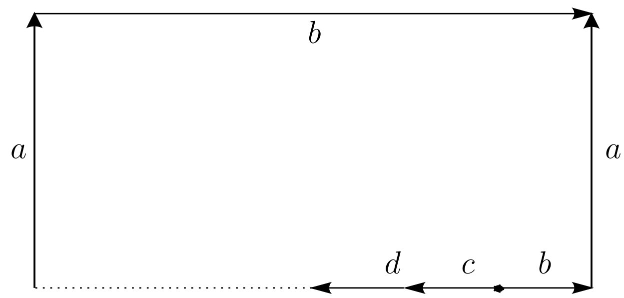

Figure 2. M_g

Part (b): Choose C' as one of the generators a of π_1 in figure on Page 5 of Chapter 0 of Hatcher. Then draw M_g as the rectangle with boundary identifications as follows:

Drawing the surface of genus g in this way, defines a *singular flat metric* on it. The retraction onto a is just the projection to a along horizontal lines. (There are other ways to construct a retraction also)