Math 525: Takehome Midterm 2 Solutions

Here is a detailed solutions for the problem that caused people the most difficulty.

Problem 3: Hatcher §2.1 #22

Let X be a finite-dimensional CW-complex. Throughout, I freely us that excision gives

$$H_n(X^k, X^{k-1}) \cong \tilde{H}_n\left(X^k/X^{k-1} \cong \bigvee_\alpha S^k\right) \cong \bigoplus_\alpha \tilde{H}_n(S^k) = \begin{cases} \bigoplus_\alpha \mathbb{Z} & \text{if } n = k \\ 0 & \text{otherwise.} \end{cases}$$

where α indexes the k-cells.

(a) *The homology group $H_i(X) = 0$ for $i > d = \dim X$, and $H_d(X)$ is free of rank at most the number of d-cells.*

Proof. If $d = 0$, then $X = X^0$ is a discrete set of points, from which (a) follows as we know $H_*(X)$ completely. Inductively, suppose (a) holds for all CW-complexes of dim $< d$. Then if X has dimension d and $i > d$, by induction the exact sequence

$$0 \cong H_i(X^{d-1}) \longrightarrow H_i(X \cong X^d) \longrightarrow H_i(X, X^{d-1}) \cong 0$$

forces $H_i(X) = 0$, as desired. For $i = d$, the exact sequence

$$0 \cong H_d(X^{d-1}) \longrightarrow H_d(X) \xrightarrow{j_*} H_i(X, X^{d-1}) \cong \bigoplus_{d\text{-cells}} \mathbb{Z}$$

implies that j_* is injective. Thus $H_d(X)$ can be regarded as a subgroup of the free abelian group $\bigoplus_{d\text{-cells}} \mathbb{Z}$, and hence is free of rank at most the number of d-cells. □

The following lemma makes (b) and (c) easier:

Lemma. *If $k > n$, then $H_n(X) \cong H_n(X^k)$.*

Proof. Since X is finite dimensional, it suffices to show $H_n(X^k) \cong H_n(X^{n+1})$ for all $k > n$. We induct on k; the base case $k = n + 1$ is trivial. Assuming it holds for some $k > n$, we have

$$0 \cong H_{n+1}(X^{k+1}, X^k) \to H_n(X^k) \to H_n(X^{k+1}) \to H_n(X^{k+1}, X^k) \cong 0$$

which forces the middle map to be an isomorphism, as desired. □

(b) *If X has no $n+1$ or $n-1$ cells, then $H_n(X)$ is free on the n-cells.*

Proof. If $n = 0$, each path component of X contains exactly one 0-cell since there are no 1-cells and ∂D^n is path connected for $n > 1$. If $n = 1$, then $X = \emptyset$ as there are no 0-cells. So assume that $n > 1$. By the lemma, $H_n(X) \cong H_n(X^{n+1} = X^n)$ as there are no $n+1$ cells. Moreover, part (a) gives us

$$0 \cong H_n(X^{n-2}) \to H_n(X^n) \to H_n(X^n, X^{n-2}) \to H_{n-1}(X^{n-2}) \cong 0$$

and as there are no $n-1$-cells, we have

$$H_n(X^n) \cong H_n(X^n, X^{n-2}) = H_n(X^n, X^{n-1}) \cong \bigoplus_{n\text{-cells}} \mathbb{Z}$$

as desired. □

(c) *If X has k n-cells then $H_n(X)$ is generated by k elements.*

Proof. By the lemma, we have $H_n(X) \cong H_n(X^{n+1})$. Moreover, exactness of

$$H_n(X^n) \xrightarrow{j_*} H_n(X^{n+1}) \to H_n(X^{n+1}, X^n) \cong 0.$$

means j_* is onto. By (a), $H_n(X^n)$ is generated by at most k elements, and hence so is the quotient group $H_n(X)$. □