Math 525: Takehome Midterm 1

Due date: In class on Wednesday, September 23.

Disclaimer, Terms, and Conditions: You may not discuss the exam with anyone except myself. You may *only* consult the following:

- The beloved(?) text, Hatcher's *Algebraic Topology*.
- Your class notes and returned HW sets.
- My online class notes and HW solutions.
- McClearly, *A first course in topology*, available at `http://math.vassar.edu/faculty/McCleary/Topos.html`.
- An abstract algebra text of your choice.

You can use any result in Hatcher in chapters 0-1, even if I didn't cover it in class. You can also use the result of any HW problem that was assigned, whether or not you did it. While I believe all the questions are stated correctly, please contact me if you think something is fishy. (I'll be out of town until Monday, Sept. 23, but I'll be checking my email regularly.)
Office hours: While I will not provide direct help on the exam problems, I will still be happy to answer questions about the course material during my usual office hours.

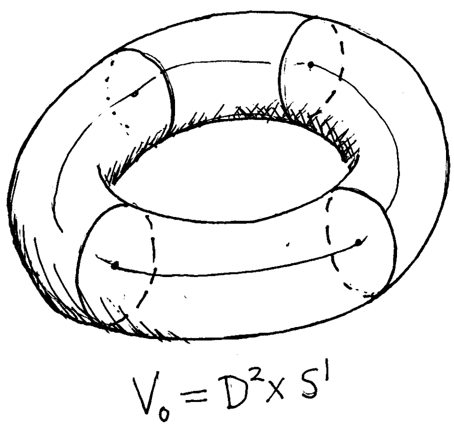
1. Suppose that $A \subset X$ is a closed subset of a Hausdorff topological space X. Suppose further that for each $x \in X \setminus A$, there are *disjoint* open sets $U, V \subset X$ with $x \in U$ and $A \subset V$.
   (a) Consider the equivalence relation on X where $x \sim x'$ if both are in A, and let $X/A = X/\sim$ have the induced quotient topology. Prove that X/A is Hausdorff.
   (b) If A is any compact subset of a metric space X, show it satisfies the above condition so that (a) applies.
2. Hatcher 1.1, page 39, problem 16.
3. Hatcher 1.3, page 79, problem 3.
4. Suppose that X is a contractible topological space. Suppose A is a retract of X. Prove that A is also contractible.
5. Show that the mapping cylinder of every map $f\colon S^1 \to S^1$ is a CW complex.

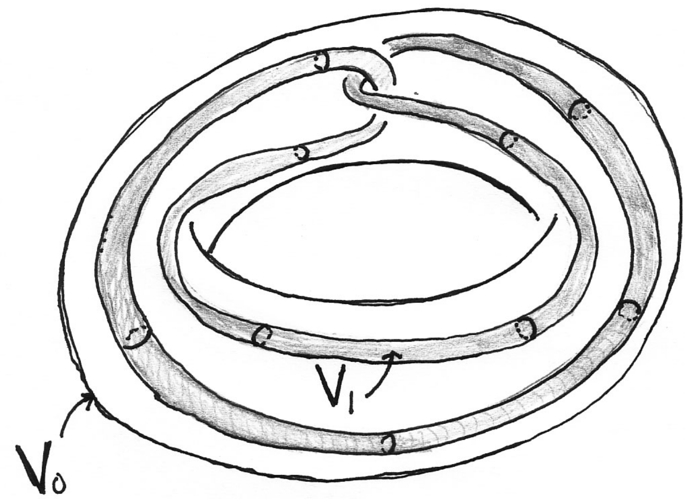
Note: There is an extra credit question, worth 1/3 a normal question, on the back!

Extra Credit Problem.

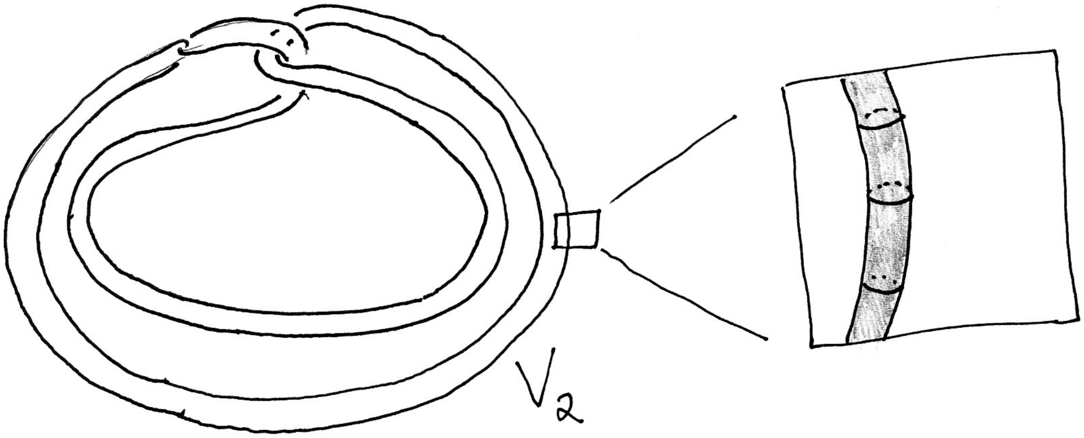
A 3-manifold is a Hausdorff topological space locally homeomorphic to $\mathbb{R}^3$ (technically, I should add 2nd countable to the definition). So $\mathbb{R}^3$ itself is one example; some compact examples are S^3, $S^2 \times S^1$, and $S^1 \times S^1 \times S^1$. In this problem, I'll construct an interesting noncompact 3-manifold, and you'll compute its fundamental group.

A *solid torus* is a space homeomorphic to $D^2 \times S^1$. Consider the a standard unknoted solid torus V_0 sitting in $\mathbb{R}^3$ as shown at left.





Let V_1 be the solid torus sitting inside V_0 as shown on the right. Note that V_1 is embedded in $\mathbb{R}^3$ in the same way as V_0, just untwist it, slide it around, and enlarge it a bit. Thus there exists a homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ with $h(V_0) = V_1$. Inductively, define $V_{i+1} = h(V_i)$. A sketch of V_2 is shown below:



1. Prove that $V_{i+1} \subset V_i$. Conclude that $X = \cap_{i=0}^{\infty} V_i$ is closed and non-empty. (In most places, X will look like the product of a Cantor set and an interval!)

2. Consider the non-compact 3-manifold $W = \mathbb{R}^3 \setminus X$. Compute its fundamental group. Hint: it's finitely generated, despite the fact that X is really complicated!