# **Math 525**: Problem Set 9[1]

**Important note:** I will be away the week of November 16. On Monday and Wednesday, Tom Nevins will be filling in for me, and there will be no class on Friday, Nov 20. I will be available by email throughout the week to answer questions about this assignment.

**Due date:** In my mailbox in Altgeld 250 by noon on *Friday*, November 20.

1. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map whose derivative $Df$ at $x$ is non-singular. As a matrix, $Df$ is just the Jacobian $(\frac{\partial f_i}{\partial x_j}(x))$. You will show that the local degree is determined by $Df$ via

$$\deg_x f = \epsilon(x) := \begin{cases} 1 & Df\colon \mathbb{R}^n \to \mathbb{R}^n \text{ preserves orientation, i.e. } \det Df > 0. \\ -1 & Df\colon \mathbb{R}^n \to \mathbb{R}^n \text{ reverses orientation, i.e. } \det Df < 0. \end{cases}$$

   Set $y = f(x)$, and let $V$ be any neighborhood of $y$. By the Inverse Function Theorem, there is a neighborhood $U$ of $x$ so that $f$ restricted to $U$ is a homeomorphism and $f(U) \subset V$.

   (a) Regard $\mathbb{R}^n$ as $S^n - \{\text{pt}\}$. If we fix a generator $\alpha \in H_n(S^n)$, show this determines generators $\overline{\alpha}$ of $H_n(U, U - x)$ and $\overline{\alpha}'$ of $H_n(V, V - y)$. Then define $\deg_x f$ by $f_*(\overline{\alpha}) = (\deg_x f)\overline{\alpha}'$. Show that this does *not* depend on the choice of $\alpha$.

   (b) Prove that $\deg_x f$ does not depend on the choice of $U$ and $V$.

   (c) Consider the 1st-order Taylor approximation $a(v) = y + (Df)(v - x)$ to $f$ at $x$. Show that $\deg_x f = \deg_0 Df$ by arguing that for some $U$ the maps $a, f\colon (U, U - x) \to (\mathbb{R}^n, \mathbb{R}^n - y)$ are homotopic, *as maps of pairs* (see page 36 of Hatcher for the definition).

   (d) If $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear map, show that $\deg_0 T$ is 1 or $-1$ depending on the sign of $\det T$. Hint: Use Gaussian elimination to show that the matrix $T$ can be joined by a path of invertible matrices to a diagonal matrix with $\pm 1$'s on the diagonal.

2. A polynomial $f(z) \in \mathbb{C}[z]$, viewed as a map $\mathbb{C} \to \mathbb{C}$, always extends to a continuous map $\hat{f}$ of the one-point compactification $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \cong S^2$.

   (a) Show that $\deg \hat{f}$ as a self-map of $S^2$ is the same as the degree of $f$ as a polynomial. One approach is to calculate $\deg_\infty \hat{f}$.

   (b) Use (a) to prove the Fundamental Theorem of Algebra.

3. Construct a surjective map $S^n \to S^n$ of degree zero, for each $n \geq 1$.

4. Hatcher, Section 2.2, #11.

5. Hatcher, Section 2.2, #13.

6. Hatcher, Section 2.2, #21.

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[1] Revised Nov. 13 to remove duplicate problems.