Math 525: Problem Set 10.[1]

Due date: In class on Wednesday, December 9.

1. Hatcher §2.2, #29. Note: This problem and the next require the very useful Meyer-Vietoris sequence, which you should read about on page 149.

2. Hatcher §2.2, #35.

3. Hatcher §2.3, #1.

4. Recall that a surface (or 2-manifold) is a Hausdorff topological space locally homeomorphic to $\mathbb{R}^2$ (technically, I should add 2nd countable to the definition). You may find it useful to know that any surface (compact or not) has the structure of a Δ-complex consisting of triangles with sides glued in pairs. This is not obvious, but a real theorem that's needed to prove the classification of compact surfaces; in any event, you can use it below.

   (a) If Σ is a connected surface, prove that $H_2(\Sigma;\mathbb{Z})$ is either 0 or $\mathbb{Z}$. (Note: don't assume that Σ is compact, or the classification of compact surfaces.)

   (b) If $H_2(\Sigma;\mathbb{Z}) = \mathbb{Z}$ then Σ is said to be *orientable*, and a choice of generator for $H_2(\Sigma;\mathbb{Z})$ is called an *orientation*. Suppose Σ_1 and Σ_2 are oriented surfaces with orientations $\alpha_i \in H_2(\Sigma;\mathbb{Z})$. Then the *degree* of a map $f\colon \Sigma_1 \to \Sigma_2$ is defined by $f_*(\alpha_1) = (\deg f)\alpha_2$. For instance, there is a map from the torus T^2 to S^2 of any degree you want. In contrast, prove that any map from S^2 to the torus T^2 has degree 0.

5. In an earlier problem, you dealt with gluings of triangles. This question will deal with the 3-dimensional case. Consider a finite collection of 3-simplices $T_1, T_2, \ldots, T_n$. Create a space X by gluing the faces of the T_i in pairs. In particular, every face of T_i is glued to precisely one face of some T_j. Prove that X is a 3-manifold if and only if $\chi(X) = 0$. Thus not every such gluing gives a 3-manifold, and indeed most don't.

   Notes: You will need to use the classification of compact surfaces in your proof. Do *not* assume the fact that odd-dimensional manifolds have Euler characteristic 0.

Extra credit problem: Let Σ be the closed orientable surface of genus 2. Does it have a self map $f\colon \Sigma \to \Sigma$ of degree 2? Prove your answer.

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[1]Revised December 8 to require the surface in #4 to be connected.