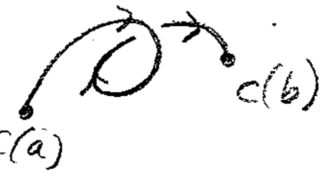
# Lecture 28: More on conservative vector fields; the curl

**Last time:** 1) $C$ a path param by $c:[a,b] \to \mathbb{R}^n$.

$f: \mathbb{R}^n \to \mathbb{R}$ differentiable.

$$\int_C \nabla f \cdot ds = f(c(b)) - f(c(a))$$




2) A vector field $F: \mathbb{R}^n \to \mathbb{R}^n$ is conservative if $F = \nabla f$ for some function $f: \mathbb{R}^n \to \mathbb{R}$.

HW: See web. Next time: Exam review.

---

$\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ a vector field is path independent if

$$\int_{C_1} \vec{F} \cdot ds = \int_{C_2} \vec{F} \cdot ds$$

for every pair of with the same endpoints.

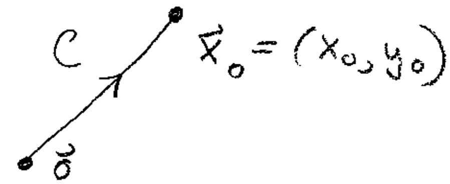

[Ex: Conservative Vector fields]

Thm: $\vec{F}$ is conservative if and only if it is path independent.

Reason: Suppose $\vec{F}$ is path independent.

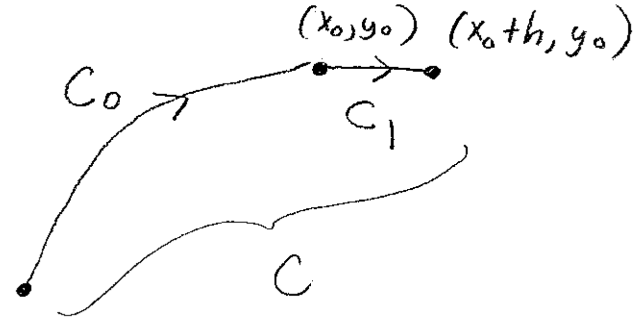
Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(\vec{x}_0) = \int_C \vec{F} \cdot ds$

for any path from $\vec{0}$ to $\vec{x}_0$.




Point: $\nabla f = \vec{F}$

Let's compute $\frac{\partial f}{\partial x}(x_0, y_0)$




$$f(x_0+h, y_0) = \int_C \vec{F} \cdot ds$$

$$= \int_{C_0} \vec{F} \cdot ds + \int_{C_1} \vec{F} \cdot ds \qquad c_1(t) = (x_0 + t, y_0), \quad 0 \le t \le h$$

$$= f(x_0, y_0) + \int_0^h F(x_0+t, y_0) \cdot (1, 0)\, dt$$

$$= f(x_0, y_0) + \int_0^h F_1(x_0 + t, y_0)\, dt \qquad \text{where } \vec{F} = (F_1, F_2)$$

So

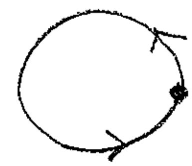
$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

$$= \lim_{h \to 0} \underbrace{\frac{1}{h} \int_0^h F_1(x_0+t, y_0)\, dt}_{\text{Average of } F(x_0+t,\, y_0) \text{ on } [0,h]} = F_1(x_0, y_0).$$

Similary $\frac{\partial f}{\partial y}(x_0, y_0) = F_2(x_0, y_0)$ and so $\vec{F} = \nabla f$.

---

<u>Note</u>: Path independence is equivalent to

$\int_C F \cdot ds = 0$ for every closed path 

---

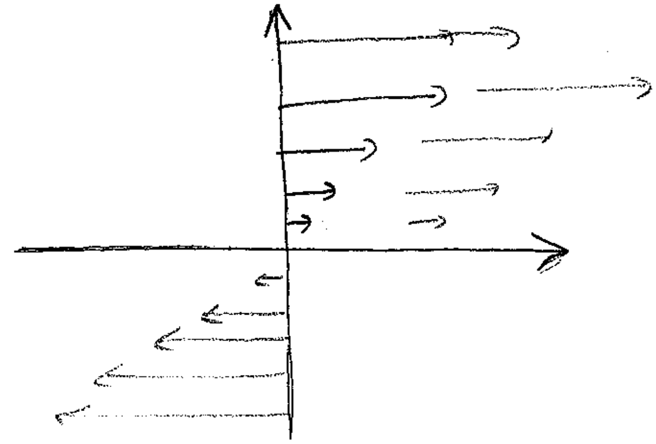
<u>Curl</u>: (§4.6) $\vec{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ a vector field, $\vec{F} = (F_1, F_2, F_3)$

$$\text{curl } F = \nabla \times F = \begin{pmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{pmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) i - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) j + \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) k$$

is another vector field.

<u>Ex</u>: $F(x, y, z) = (y, 0, 0)$

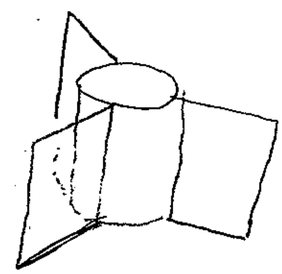


$$\text{curl } F = \begin{pmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & 0 & 0 \end{pmatrix}$$

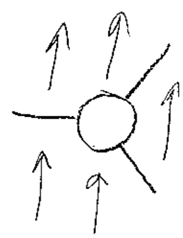
$$= (0, 0, -1)$$

# What does the curl measure?

First consider a vector field just in $\mathbb{R}^2$:



Place a small paddle-wheel into the flow



In this case $|\text{curl } F|$

Note: moves along with the flow

is the rate of rotation [more precisely 2(angular velocity)] of the paddle, and curl F points in the positive z-direction if the rotation is counter-clockwise.

Note: For $F: \mathbb{R}^2 \to \mathbb{R}^2$ if we consider the corresponding vector field on $\mathbb{R}^3$ given by

$\overline{F} = (F_1(x,y), F_2(x,y), 0)$, then

$$\text{curl } \overline{F} = \Big(0, 0, \underbrace{\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}}_{\text{scalar curl}}\Big)$$

[What has this got to do with conservative vector fields?]

Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ and $\vec{F} = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

Then scalar curl$(F) = \frac{\partial f}{\partial x \partial y} - \frac{\partial f}{\partial y \partial x} = 0.$

Similarly, if $f: \mathbb{R}^3 \to \mathbb{R}$ then curl$(\nabla f) = \vec{0}$.

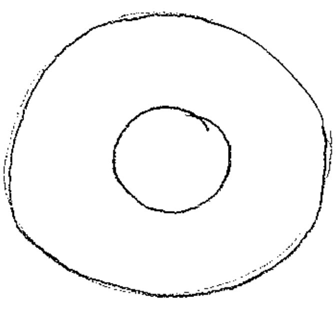
So if $\vec{F} = (y, 0)$, then scalar curl $= -1$

$\Rightarrow \vec{F}$ is not conservative.

<u>Q</u>: Suppose $\vec{F}$ is a vector field defined on some region $U$ of $\mathbb{R}^2$. If scalar curl $\vec{F} = 0$, must $F$ be conservative?

<u>A</u>: Yes if $U = \mathbb{R}^2$ or if $U$ "has no holes." (<u>simply connected</u>)

<u>Ex</u>: $U = \{1 < \|\vec{x}\| < 2\}$



$$F(x, y) = \frac{1}{\sqrt{x^2 + y^2}}(-y, x)$$

has scalar curl $= 0$

but is not $= \nabla f$ for any fn $f: U \to \mathbb{R}$.

<u>Why?</u> Well

$$\text{scalar curl } F = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

$$= \frac{(x^2+y^2)^{1/2} + x(x^2+y^2)^{-1/2}}{(x^2+y^2)}$$