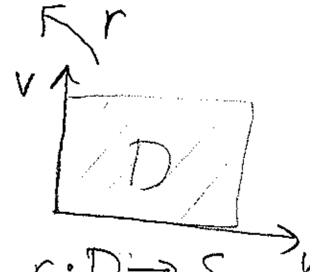
Lecture 38: More on integration over surfaces. (§7.3, 7.4)

HW: Due Tuesday April 8: §7.3: 16, 17

Next time: Rest of Chapter 7. Office hours today end @ 4:00.

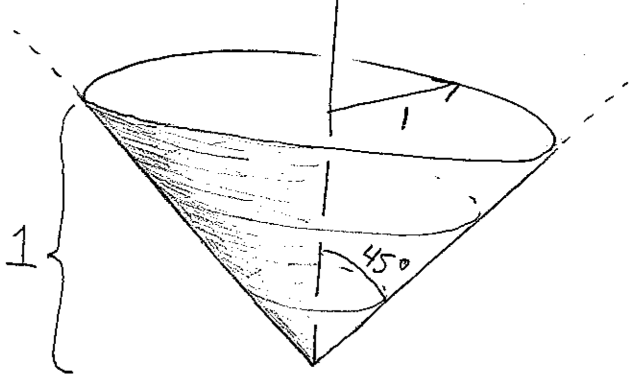
Last time: For a surface S in $\mathbb{R}^3$ and a function $f: \mathbb{R}^3 \to \mathbb{R}$

$$\iint_S f \, dA = \iint_D f \circ r(u,v) \, \|T_u \times T_v\| \, du \, dv$$

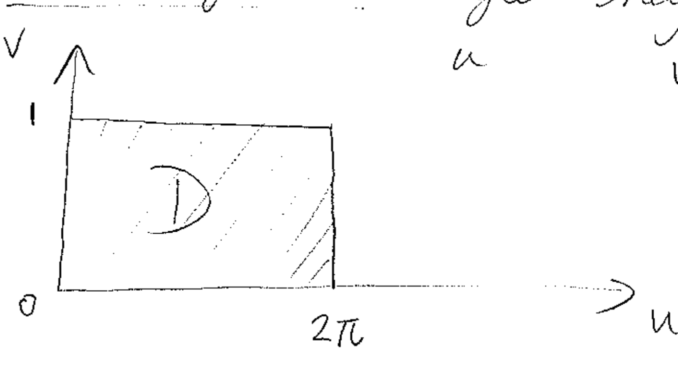


$r: D \to S$ a parameterization

Ex: Find the average of $f(x,y,z) = xy + z$ on the cone S given by $x^2 + y^2 = z^2$ for $0 \le z \le 1$.

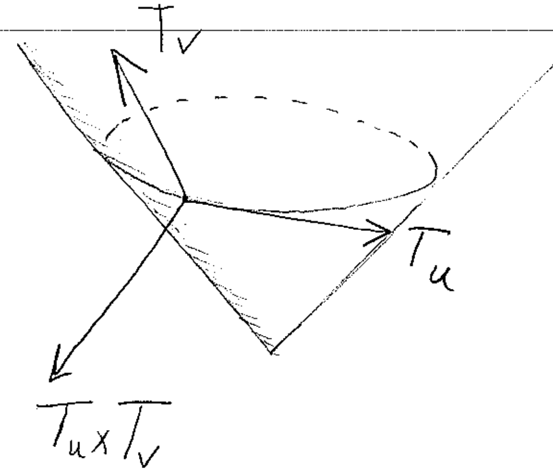


Parameterization: angle u, height v

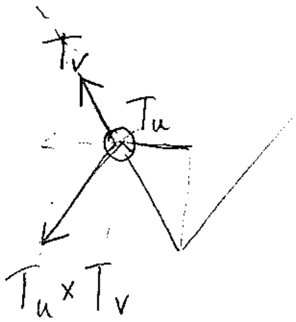


$$r(u,v) = (v\cos u, v\sin u)$$

$$T_u = (-v\sin u, v\cos u, 0)$$
$$T_v = (\cos u, \sin u, 1)$$



$$T_u \times T_v = (v\cos u, v\sin u, -v)$$

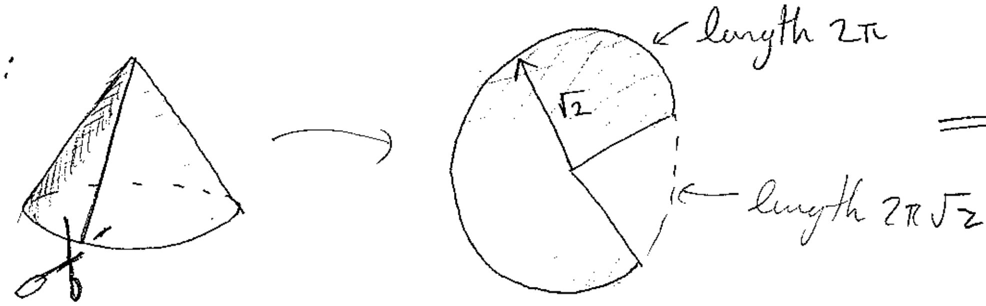


which has length $\sqrt{2}\, v$.

Area: $\iint_S 1\, dA = \iint_D \|T_u \times T_v\|\, du\, dv$

$$= \int_0^1 \int_0^{2\pi} \sqrt{2}\, v\, du\, dv = \int_0^1 2\sqrt{2}\pi v\, dv = \sqrt{2}\pi$$

Check:



$\Longrightarrow$ Area of cone $= \frac{1}{\sqrt{2}}$ area inside circle

$$= \frac{1}{\sqrt{2}}(\pi \sqrt{2}^2)$$
$$= \sqrt{2}\pi \checkmark$$

Integral: $\iint_S xy + z\, dA$

$$= \int_0^1 \int_0^{2\pi} (v^2 \underbrace{\sin u \cos u}_{\frac{1}{2}\sin 2u} + v)(\sqrt{2}\, v)\, du\, dv = \int_0^1 2\sqrt{2}\pi v^2\, dv$$

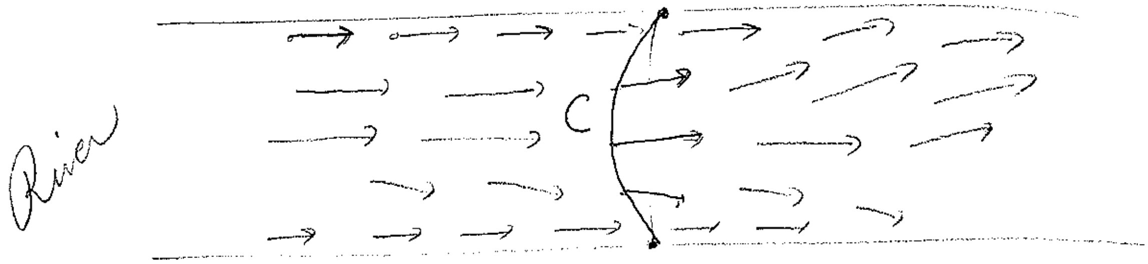
$$= \frac{2\sqrt{2}\pi}{3}$$

Average: $\boxed{\frac{2}{3}}$

Integrating vector fields.

Back to 2d $\quad F: \mathbb{R}^2 \to \mathbb{R}^2$ a vector field

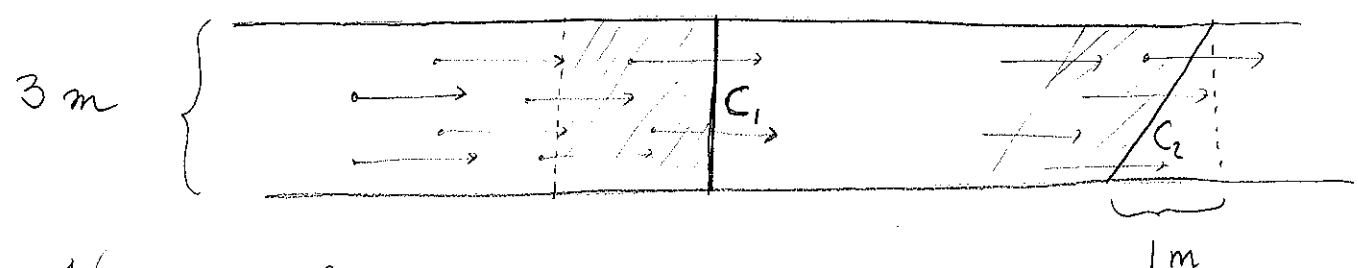
represents a fluid flow.



Q: What is the rate that water is crossing C? (the flux.)

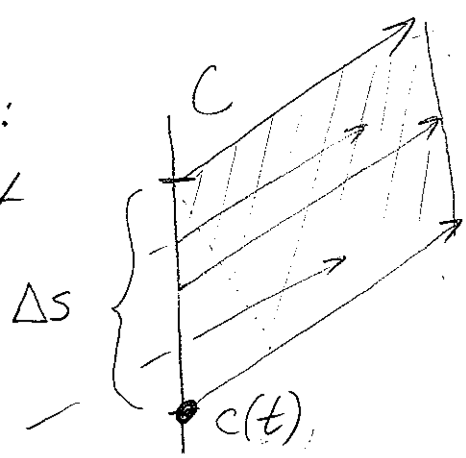
Note the answer has units $(\text{area})/(\text{time})$ since we are just looking at the surface flow.

Ex: $F = 2j = (2\,m/s, 0, 0)$



$\text{Flux}_{C_1} = 6\ m^2/s \qquad \text{Flux}_{C_2} = 6\ m^2/s$

Closeup for a general curve:
The flux across this segment of length Δs is the shaded area

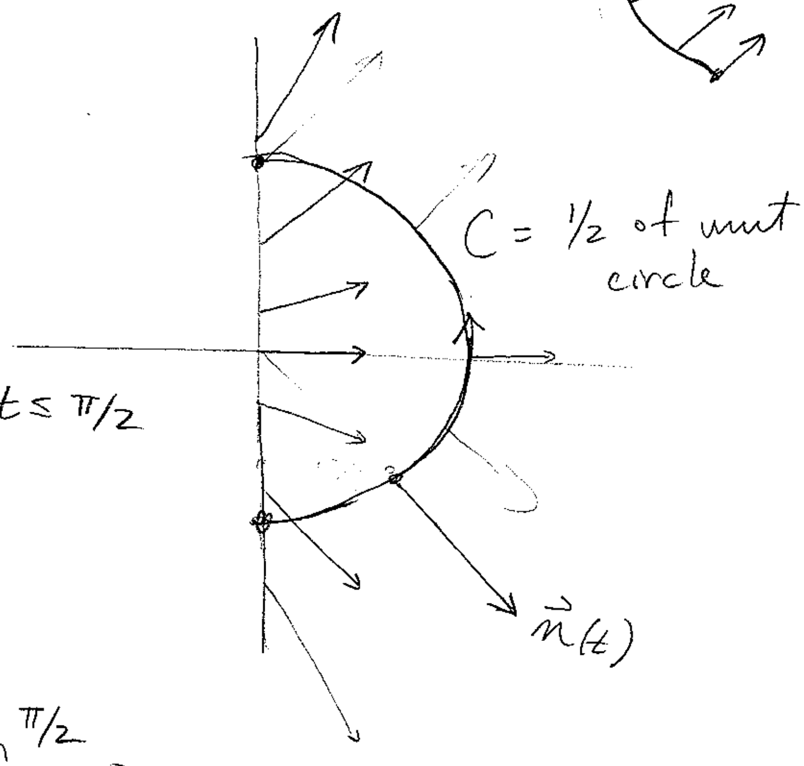


which is $\| (\Delta s \, c'(t) \times \vec{F}(c(t)) \|$ or equivalently

$= (\vec{F} \cdot n) \, \Delta s$ where n is a unit normal vector to C

<u>Ex</u>: $F = (1, y)$

what is flux across C?



Parameterize

$C(t) = (\cos t, \sin t) \quad -\pi/2 \le t \le \pi/2$

$n(t) = (\cos t, \sin t)$

<u>Flux</u>:

$$\int_C (\vec{F} \cdot \vec{n}) \, ds = \int_{-\pi/2}^{\pi/2} \vec{F}(c(t)) \cdot n(t) \, dt$$

$$= \int_{-\pi/2}^{\pi/2} (1, \sin t) \cdot (\cos t, \sin t) \, dt = \int_{-\pi/2}^{\pi/2} \cos t + \sin^2 t \, dt$$

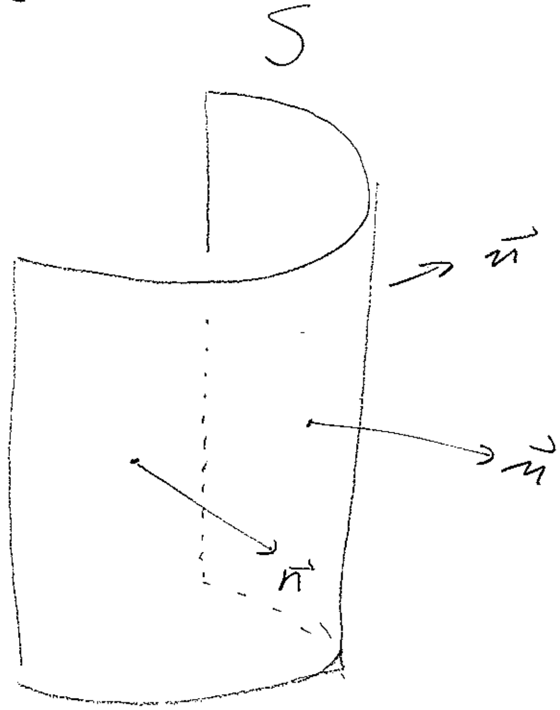
$$= 2 + \pi/2 .$$

Now lets generalize to $\mathbb{R}^3$

The flux across S is

$$\iint_S (\vec{F} \cdot \vec{n})\, dA$$

where $\vec{n}$ is a unit normal. This is also denoted $\iint_S F \cdot dA$.



Ex: