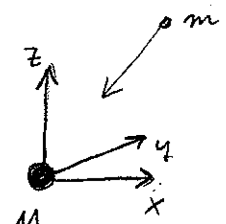
Lecture 27: Conservative Vector Fields (§5.4)

Last time: C a curve in $\mathbb{R}^n$ parameterized by $c:[a,b]\to\mathbb{R}^n$

$f:\mathbb{R}^n\to\mathbb{R}$, $F:\mathbb{R}^n\to\mathbb{R}^n$

$$\int_C f\,ds = \int_a^b f(c(t))\,\|c'(t)\|\,dt; \quad \int_C F\cdot ds = \int_a^b F(c(t))\cdot c'(t)\,dt$$

Next time: More on §5.4.



Heavy mass M at $\vec{0}$ in $\mathbb{R}^3$, small mass m at $\vec{r}=(x,y,z)$. [Then Newton says] the force on m is

$$\vec{F} = -\frac{GMm}{\|\vec{r}\|^2}\frac{\vec{r}}{\|\vec{r}\|} \qquad [G = \text{grav constant.}]$$

The gravitational potential of m is (= potential energy)

$$V = -\frac{GMm}{\|\vec{r}\|} = -GMm(x^2+y^2+z^2)^{-1/2}$$

[Note, this is larger the farther you are from $\vec{0}$.]

Key: $\vec{F} = -\nabla V$

Check: $-\nabla V = GMm\left(-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2x,\ -\frac{1}{\|\vec{r}\|^3}y,\ -\frac{1}{\|\vec{r}\|^3}z\right)$

$$= -\frac{GMm}{\|\vec{r}\|^3}\vec{r}. \checkmark$$

[In other words, the force tries to decrease the potential energy. Similar setup if we had two charged particles.]

A vector field $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ is called <u>conservative</u> or a <u>gradient vector field</u> if $\vec{F} = \nabla f$ for some $f: \mathbb{R}^n \to \mathbb{R}$. [Here $f = -V$ in our example.]

[Why the name?]

<u>Conservation Law</u>: Consider motion of smaller mass $\vec{r}(t)$

$$E(t) = (\text{kinetic}) + (\text{potential}) = \frac{1}{2} m \left(\vec{r}\,'(t) \cdot \vec{r}\,'(t)\right) + V(\vec{r}(t))$$

$$E'(t) = m\, \vec{r}\,''(t) \cdot r'(t) + \nabla V(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$$= \underbrace{\left(m\, \vec{r}\,''(t) - \vec{F}(r(t))\right)}_{= 0 \text{ by Newton's } 2^{nd} \text{ Law}} \cdot \vec{r}\,'(t) = 0$$

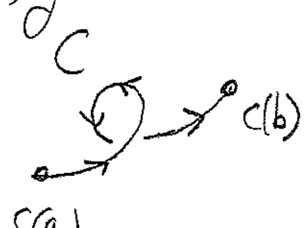
<u>Q</u>: Which vector fields are conservative? [Not all are...]

<u>Recall</u>: Fundamental Theorem of Calculus

$f: [a,b] \to \mathbb{R}$ a diff. function

Then $\displaystyle\int_a^b f'(x)\, dx = f(b) - f(a)$

<u>Thm</u>: $f: \mathbb{R}^n \to \mathbb{R}$ a diff fn, C in $\mathbb{R}^n$ a curve param by $c: [a,b] \to \mathbb{R}^n$. Then



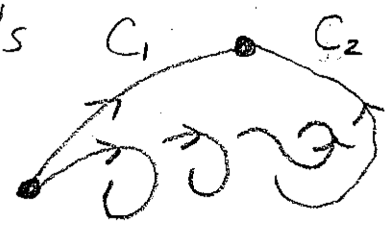
$$\int_C \nabla f \cdot ds = f(c(b)) - f(c(a))$$

<u>Reason:</u>

$$\int_C \nabla f \cdot ds = \int_a^b \nabla f(c(t)) \cdot c'(t)\, dt = \int_a^b (f \circ c)'(t)\, dt$$

$$= f(c(b)) - f(c(a)).$$

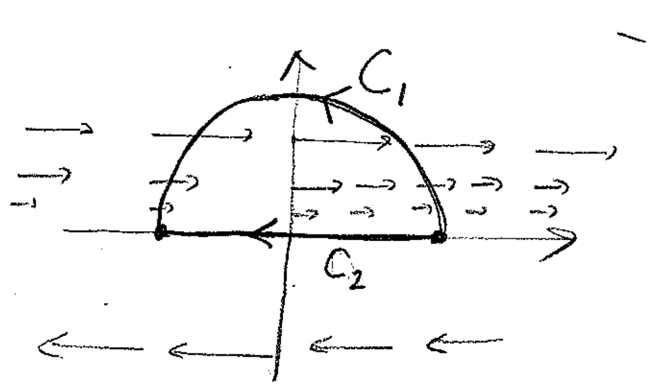
<u>Consequence:</u> If $\vec{F} = \nabla f$ then $\int_{C_1} F \cdot ds = \int_{C_2} F \cdot ds$ for any to paths with the same endpoint (independence of path)



——— ∘ ———

Here's a vector field which is <u>not</u> conservative:

$$\vec{F}(x,y) = (y, 0)$$



$$\int_{C_2} \vec{F} \cdot ds = \int_{C_2} \vec{0} \cdot c_2'(t)\, dt$$
$$= 0.$$

$c_1(t) = (\cos t, \sin t) \quad 0 \le t \le \pi.$

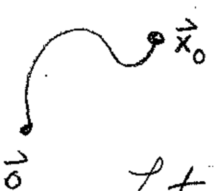
$c_1'(t) = (-\sin t, \cos t)$

$$\int_{C_1} \vec{F} \cdot ds = \int_0^{\pi} (\sin t, 0) \cdot (-\sin t, \cos t)\, dt = \int_0^{\pi} -\sin^2 t\, dt = -\pi/2$$

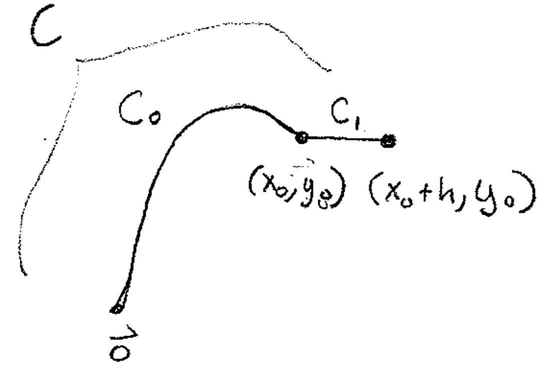
So $\int_C \vec{F} \cdot ds$ is not independent of path, hence $\vec{F}$ is not conservative. ——— ∘ ———

[In fact, independence of path implies that $\vec{F}$ is conservative;] If $\int_C F \cdot ds$ is independent of path, set

$$f(\vec{x}_0) = \int_C F \cdot ds \quad \text{where } C \text{ is a path from } \vec{0} \text{ to } \vec{x}_0$$



Point: $\nabla f = \vec{F}$.



Lets compute $\frac{\partial f}{\partial x}(x_0, y_0)$

$$f(x_0+h, y_0) = \int_C \vec{F} \cdot ds = \int_{C_0} \vec{F} \cdot ds + \int_{C_1} \vec{F} \cdot ds$$

$$= f(x_0, y_0) + \int_0^h \vec{F}(x_0+t, y_0) \cdot (1,0)\, dt$$

$$= f(x_0, y_0) + \int_0^h F_1(x_0+t, y_0)\, dt \qquad \vec{F} = (F_1, F_2)$$

as $c_1 : [0,h] \to \mathbb{R}^2$ and $c_1(t) = (x_0+t, y_0)$. So

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} = \lim_{h \to 0} \underbrace{\frac{1}{h}\int_0^h F_1(x_0+t, y_0)\, dt}_{\substack{= \text{average value of} \\ F(x_0+t,\, y_0) \text{ on } [0,h]}}$$

$$= F(x_0, y_0).$$