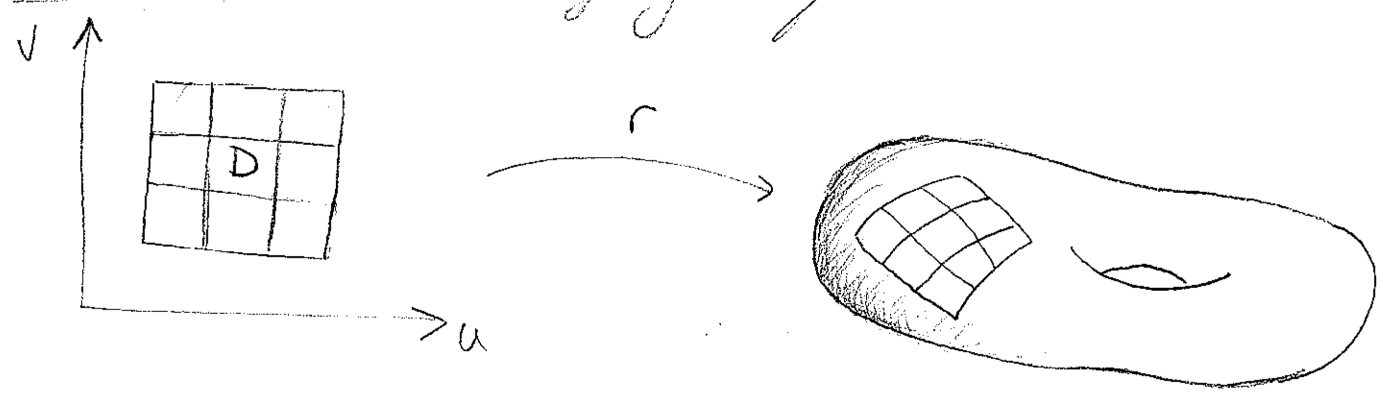
Lecture 37 Surface Area and Integration on Surfaces

HW: From printout.

Next time: More on 7.3, 7.4.

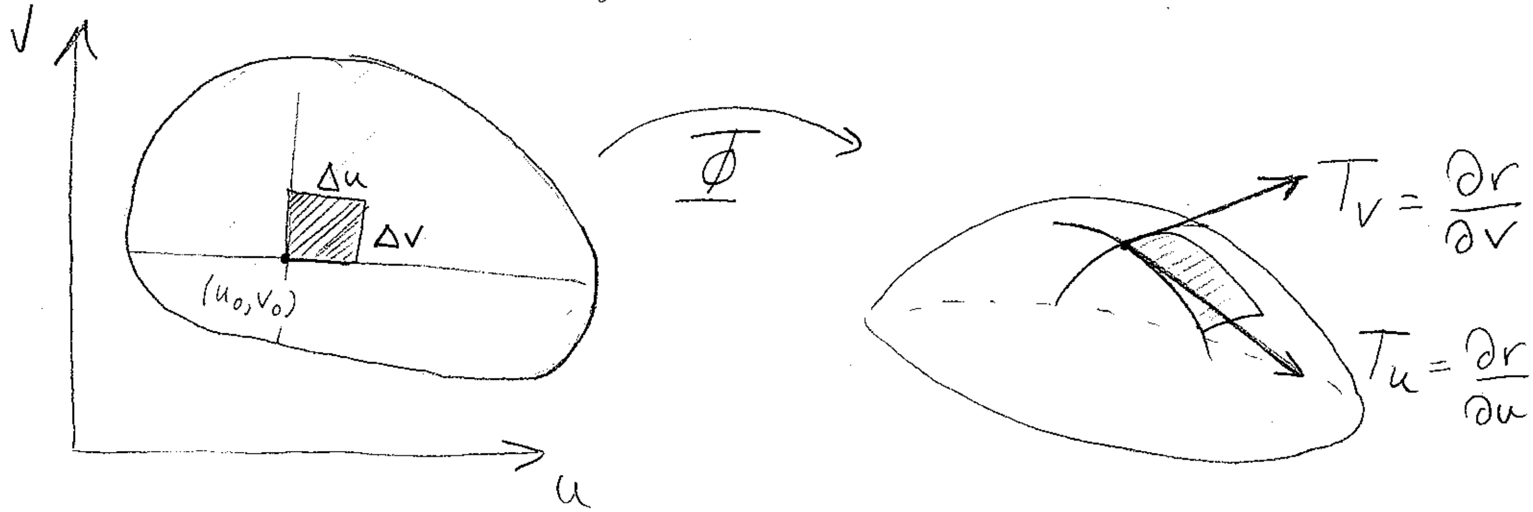
Schey: Div, grad, curl and all that.

Last time: Parametrizing surfaces

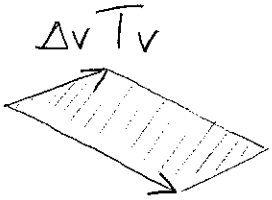


$$r : (D \text{ in } \mathbb{R}^2) \longrightarrow (S \text{ in } \mathbb{R}^3)$$

Q: What is the area of a surface?

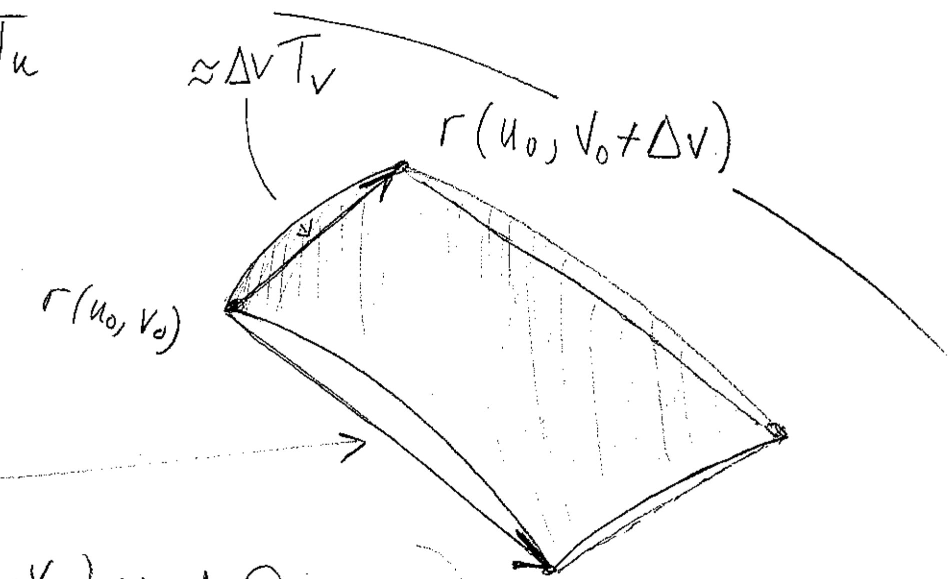


We can approximate the area of region at right by



the parallelogram shown

as you can see from this → diagram.



$$r(u_0+\Delta u, v_0) - r(u_0, v_0) \approx \Delta u \frac{\partial r}{\partial u}(u_0, v_0)$$
$$= \Delta u T_u$$

The area of this parallelogram is

$$\| (\Delta u T_u) \times (\Delta v T_v) \| = \| T_u \times T_v \| \Delta u \Delta v.$$

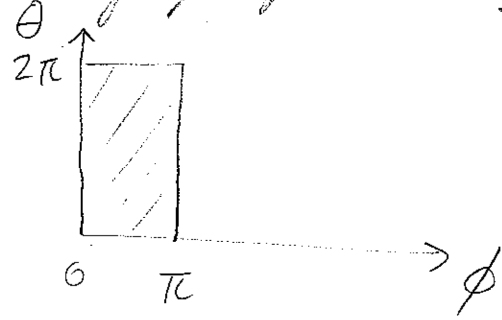
Thus the area of $S \approx \sum_{\substack{\text{small} \\ \text{squares}}} \| T_u \times T_v \| \Delta u \Delta v$

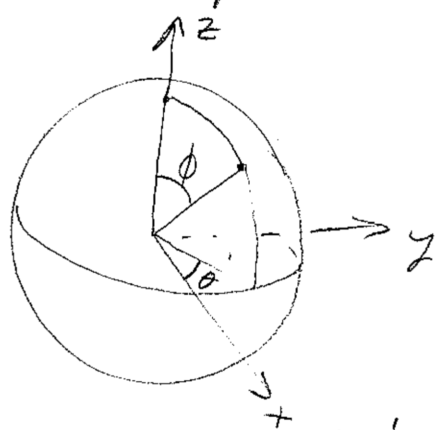
which implies

$$\text{Area}(S) = \iint_D \| T_u \times T_v \| \, du \, dv$$

[Can think of $\|T_u \times T_v\|$ as a change of area factor, just as in 2d change of variables
[Cf. Distortion of area in map projections.]]

Ex: Unit sphere $D =$





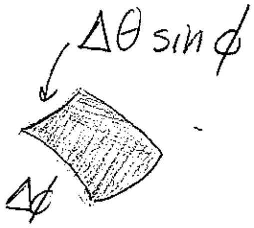
$$r(\phi,\theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

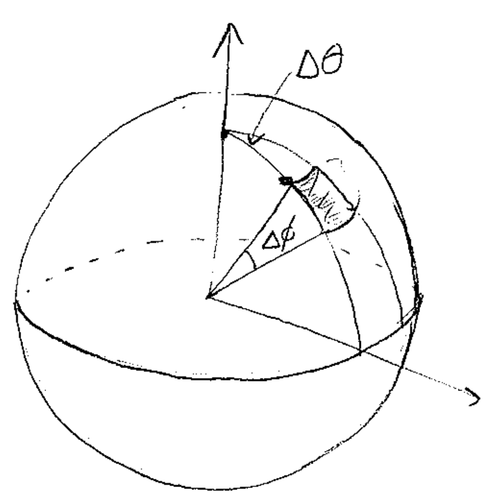
$$T_\phi \times T_\theta = \det\begin{pmatrix} i & j & k \\ — & T_\phi & — \\ — & T_\theta & — \end{pmatrix}$$

$$= (\sin^2\phi\cos\theta, -\sin^2\phi\sin\theta, \sin\phi\cos\phi)$$

$$\|T_\phi \times T_\theta\| = \sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi} = \sin\phi$$

Geometrically, we saw this when we were messing with spherical coordinates.

since  has area $\approx \sin\phi \, \Delta\phi \, \Delta\theta$.

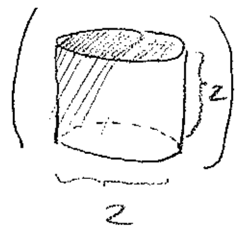


Either way, we have

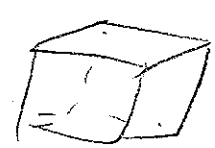
Area () $= \iint_D \sin\phi \, d\phi \, d\theta$

$$= \int_0^{2\pi} \int_0^{\pi} \sin\phi \; d\phi \, d\theta = \int_0^{2\pi} -\cos\phi \Big|_{\phi=0}^{\phi=\pi} d\theta$$

$$= \int_0^{2\pi} 2 \, d\theta = 4\pi$$

<u>Aside:</u> Area () $= 2$ Area () $+ 4\pi = 6\pi$

So $\dfrac{\text{Area}(\text{cylinder})}{\text{Area}(\text{sphere})} = \dfrac{3}{2} = \dfrac{\text{Volume}(\text{cylinder})}{\text{Area}(\text{sphere})}$

This isn't typical, e.g. compare a cube 

and a sphere.

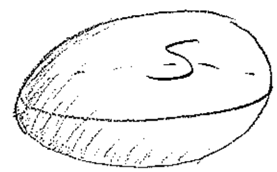
With curves, finding

$$\text{arc length} = \int_C ds = \int_a^b \|c'(t)\|\,dt$$

lead too integrating a function $f: \mathbb{R}^3 \to \mathbb{R}$

$$\int_C f\,ds = \int_a^b f(c(t))\,\|c'(t)\|\,dt$$

Similarly, if $f: \mathbb{R}^3 \to \mathbb{R}$

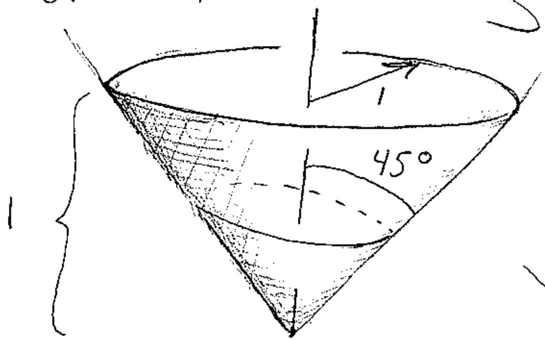


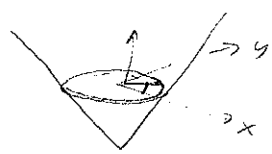
$$\iint_S f\,dA = \iint_D f \circ r(u,v)\,\|T_u \times T_v\|\,du\,dv$$

where $r: D \to S$ is a parametrization.

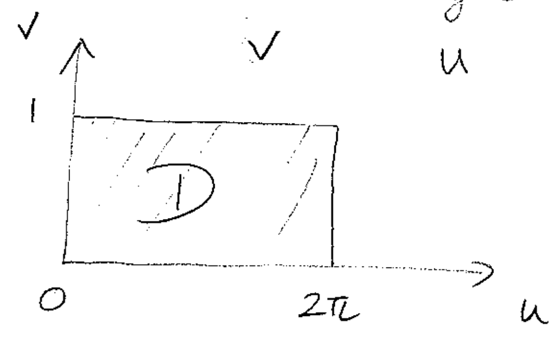
Ex: Find the average of $f(x,y,z) = xy + z$

over the cone S given by $x^2+y^2=z^2$, $0 \le z \le 1$.





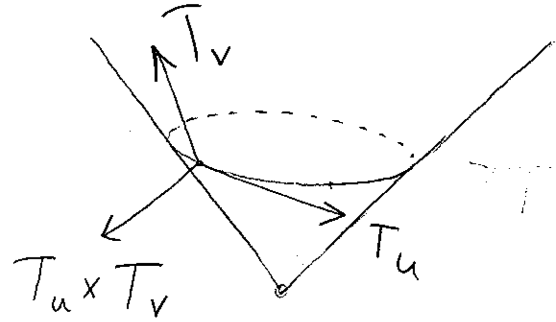
Parametrization: height v, angle u



$$r(u,v) = (v\cos u,\ v\sin u,\ v)$$

$$T_u = (-v\sin u,\ v\cos u,\ 0)$$

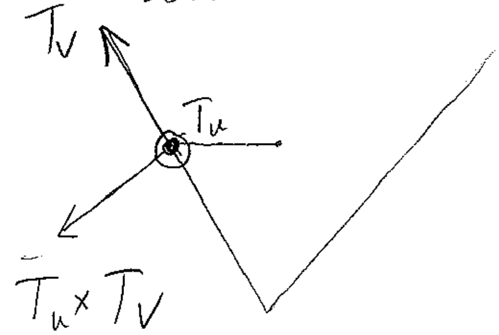
$$T_v = (\cos u,\ \sin u,\ 1)$$



$$T_u \times T_v = (v\cos u,\ v\sin u,\ -v)$$

which has length $\sqrt{2}\,v$

Cross section



Area: $\iint_S 1\, dA =$

$$\iint_D \sqrt{2}\,v\, du\, dv = \int_0^1\int_0^{2\pi} \sqrt{2}\,v\, du\, dv$$

$$= \int_0^1 2\sqrt{2}\,\pi\, v\, dv = \sqrt{2}\,\pi.$$