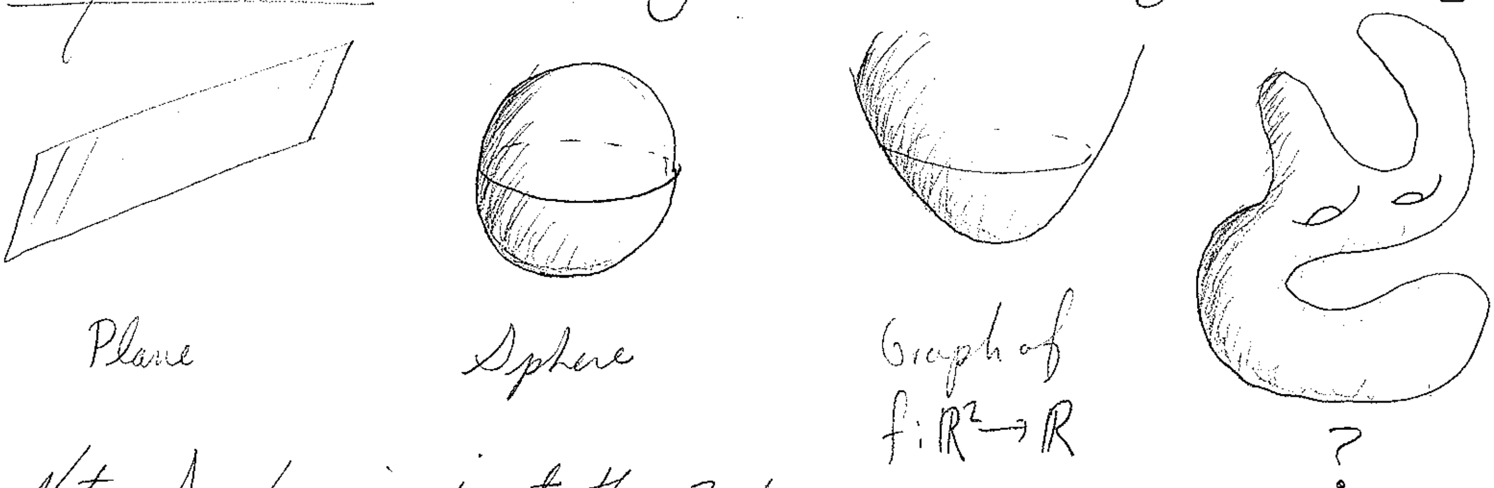
Lecture 36: Surfaces in $\mathbb{R}^3$ (§7.1–7.2)

HW: 7.1 # 3, 9, 12, 13

Next time: § More on 7.1-2. Office hours today from 4–5:30.

Surfaces in $\mathbb{R}^3$ [Something which looks locally like $\mathbb{R}^2$]

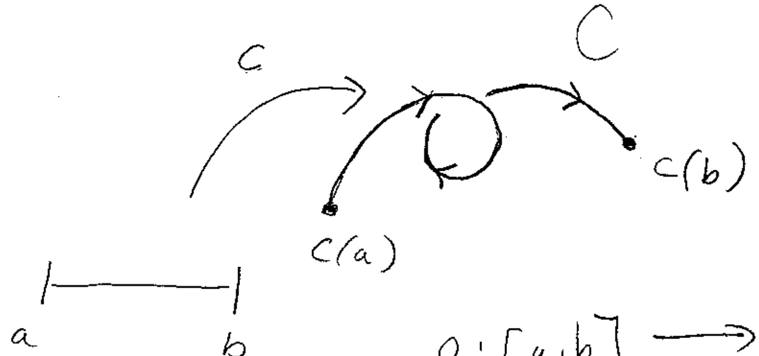


Note: Surface is just the 2-d part, not the inside.

[Where course is going: integration over surfaces and the relationship to integration over curves]

Parameterization:

Curves:



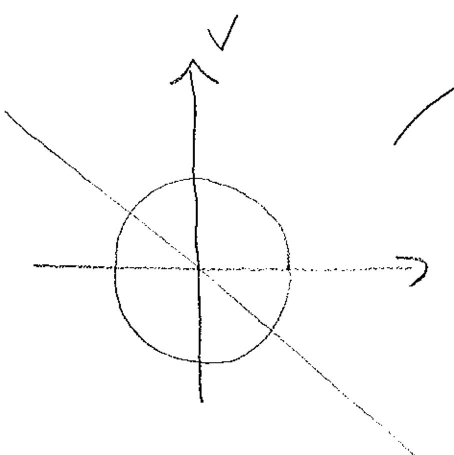
$c: [a,b] \longrightarrow \mathbb{R}^3$

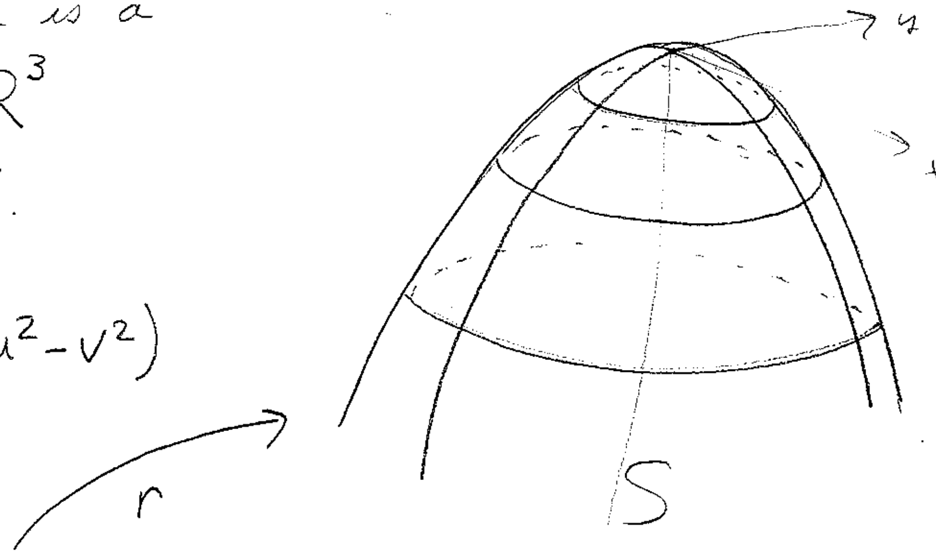
Surfaces:

Ex 1: S = graph of $f(x,y) = -x^2 - y^2$

Parameterization is a
$r: \mathbb{R}^2 \to \mathbb{R}^3$
with image S.

$$r(u,v) = (u, v, -u^2 - v^2)$$



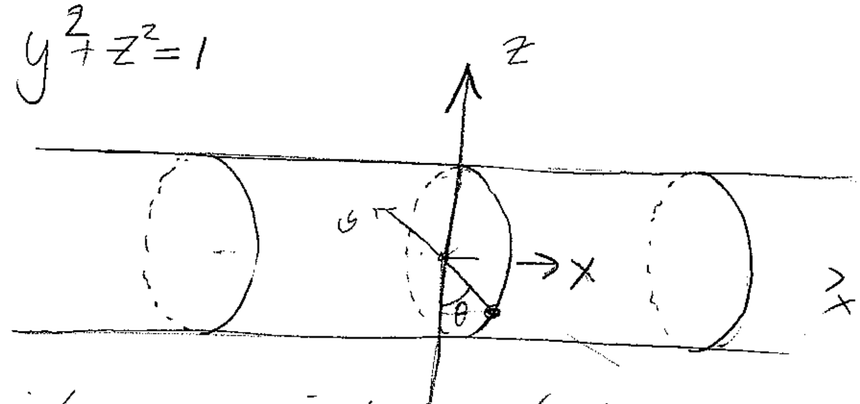


Note: A similar thing works for the graph of any function

$$r(u,v) = (u, v, f(u,v))$$

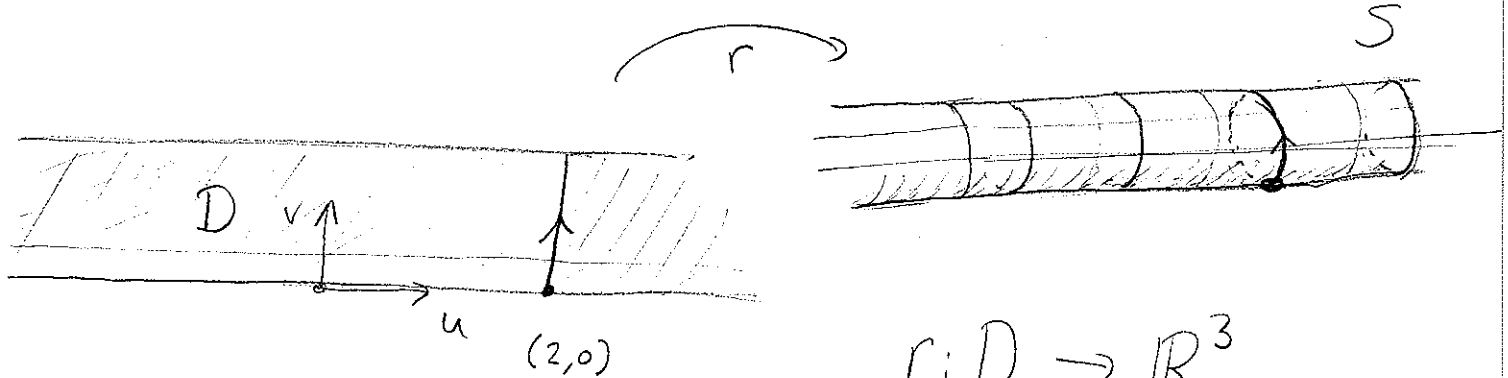
Ex 2:

S = cylinder given by $y^2 + z^2 = 1$



To specify a point need to give
x coor $\to u$
θ angle $\to v$

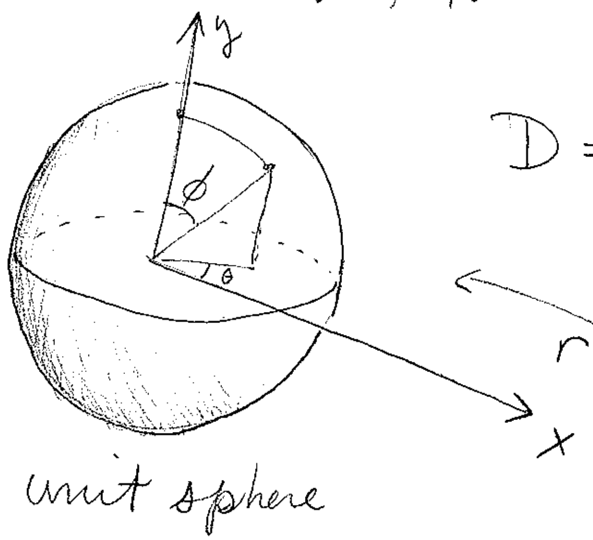
$D = \{0 \leq v \leq 2\pi\}$ $\qquad r(u,v) = (u, -\sin v, -\cos v)$



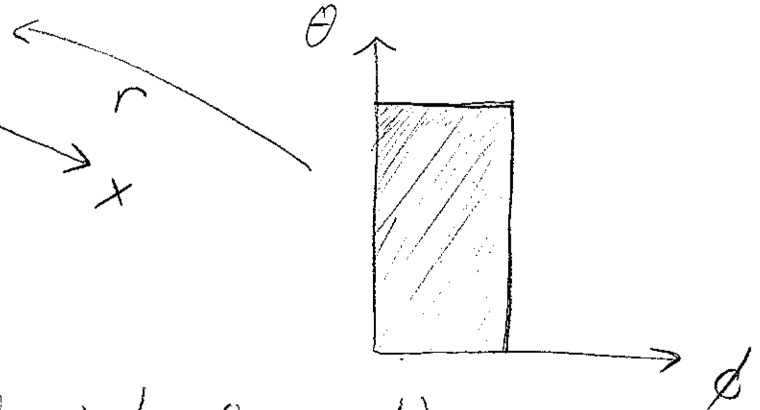
$r : D \to \mathbb{R}^3$

$r(2,0) = (2, 0, -1)$

<u>Ex 3:</u>



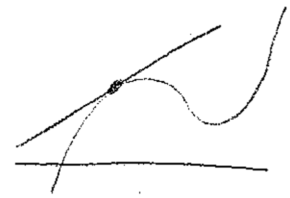
$$D = \left\{ \begin{array}{l} 0 \leq \phi \leq \pi \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$$

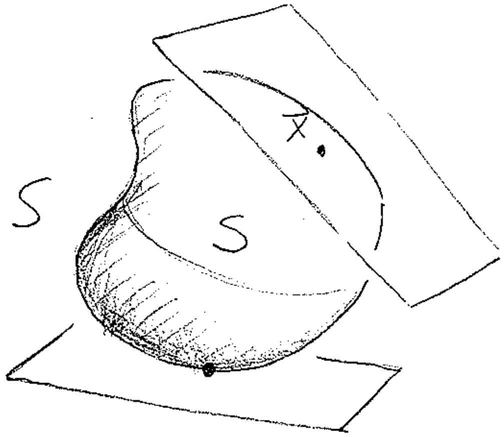


$$r(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

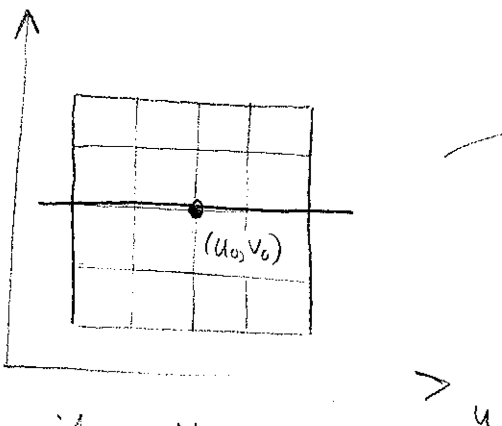
[Sections 7.1 and 7.2 give many more examples.]

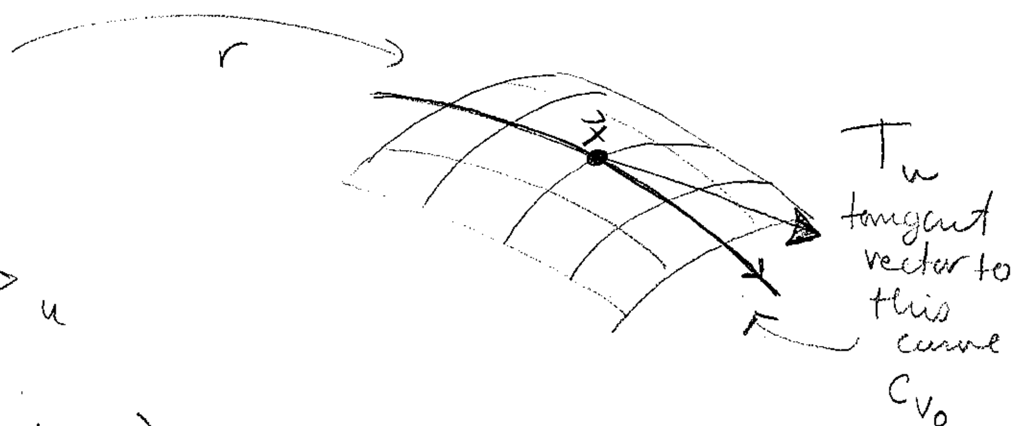
Tangent plane

at a point $\vec{x}$ in S is the plane which locally approximates S near $\vec{x}$. [Compare ]



Consider a param $r : D \to S$





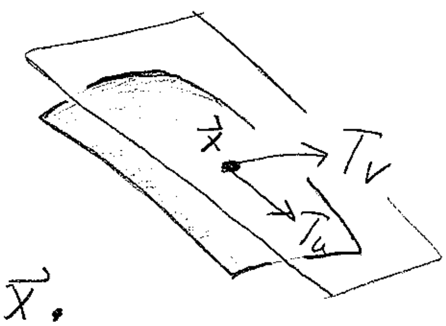
Consider the curve

$$c_{v_0}(t) = r(u_0 + t, v_0)$$

which passes through $\vec{x}$ at time $t = 0$. Its tangent vector at $t = 0$ is called

$$T_u(u_0, v_0) = c'_{v_0}(0) = \frac{\partial r}{\partial u}(u_0, v_0)$$

Similarly, we can talk about T_v



T_u and T_v span the tangent plane at $\vec{x}$.

<u>Ex</u>: Find the tangent plane to the unit sphere at $(1/\sqrt{2}, 0, 1/\sqrt{2})$. In terms of

$$r(\phi, \theta) = (\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi)$$

this point is $r(\pi/4, 0)$

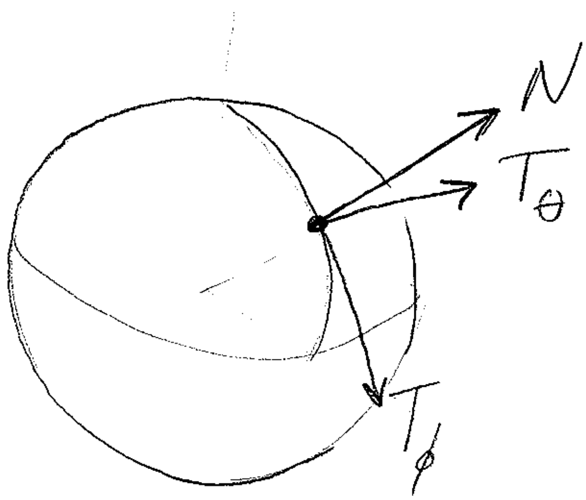
$$T_\phi = \frac{\partial r}{\partial \phi} = (\cos\phi\cos\theta,\ \cos\phi\sin\theta,\ -\sin\phi)$$

$$T_\theta = \frac{\partial r}{\partial \theta} = (-\sin\phi\sin\theta,\ \sin\phi\cos\theta,\ 0)$$

At $(\pi/4, 0)$ these are $T_\phi = (1/\sqrt{2}, 0, -1/\sqrt{2})$

$$T_\theta = (0, 1/\sqrt{2}, 0)$$

So a normal vector is $N = T_\phi \times T_\theta = (1/2, 0, 1/2)$



which points in the direction we expect.