Lecture 35: Change of variables in 3d (§6.5)

HW: §6.5: # 24, 29, 31

Next time: Starting Ch 7.

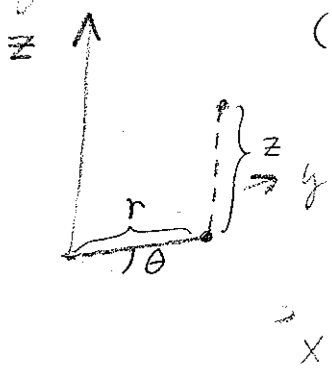
Last time:

Cylindrical Coordinates

(r, θ, z)

$0 \le r$

$0 \le \theta \le 2\pi$

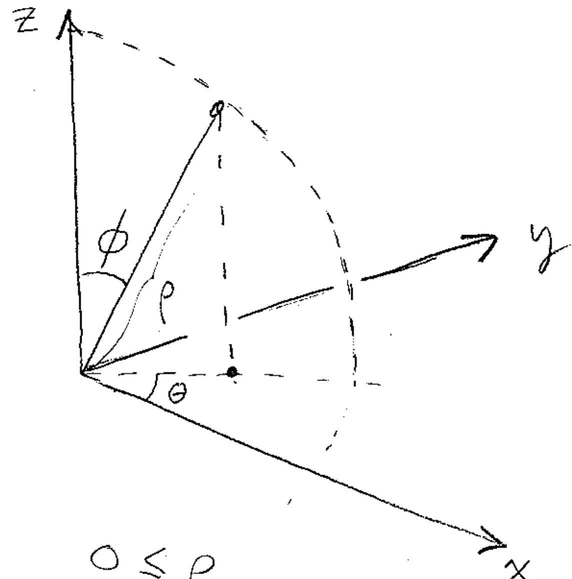


$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

Spherical Coordinates (ρ, ϕ, θ)

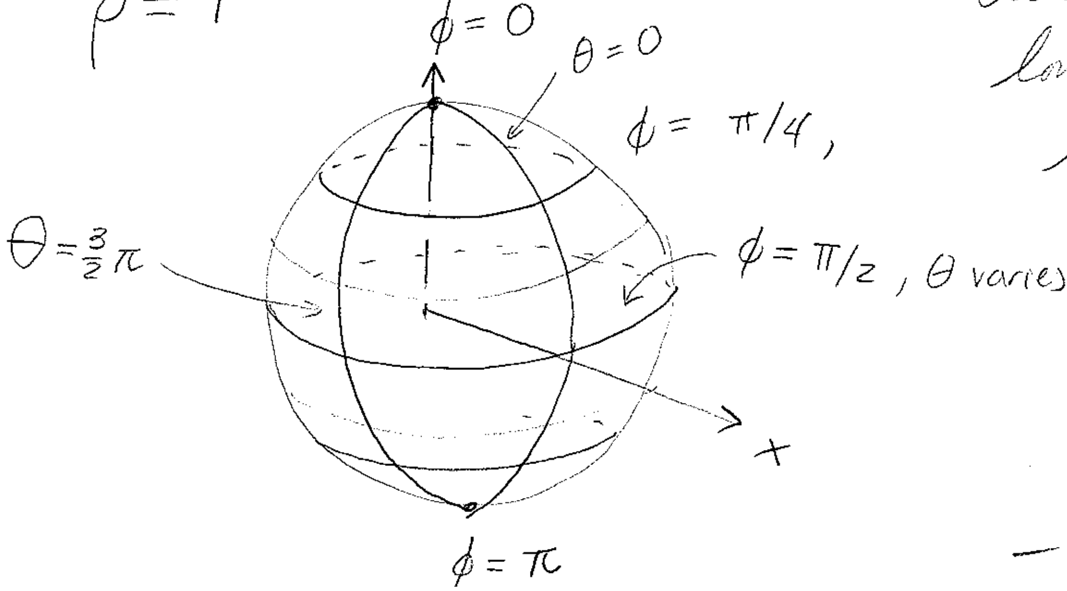


$0 \le \rho$

$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

$\rho = 1$



Like latitude – longitude, except that in map we use

instead where

$\psi = \phi - \pi/2$

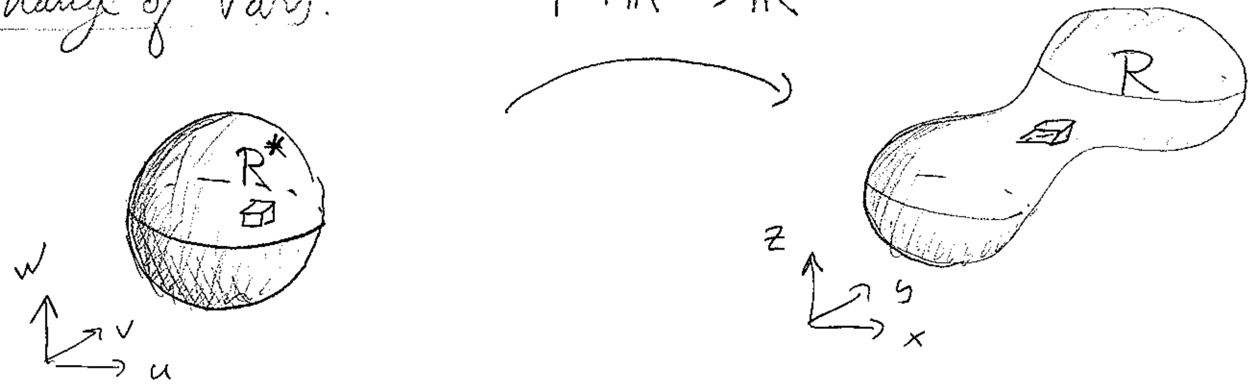
$-\pi/2 \le \psi \le \pi/2$

$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$

$0 \le \rho \le 1$
$0 \le \phi \le \pi$
$0 \le \theta \le 2\pi$

Need to do integrals in these coordinates.

<u>Change of Vars:</u> $\quad T: \mathbb{R}^3 \to \mathbb{R}^3 \quad f: \mathbb{R}^3 \longrightarrow \mathbb{R}$

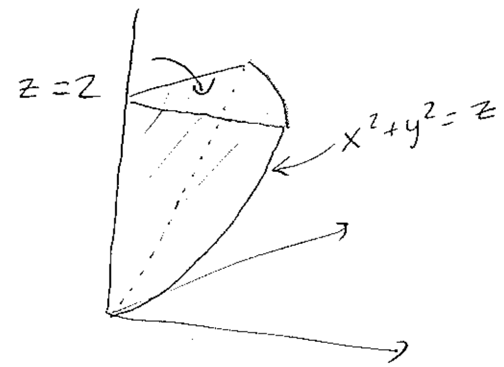


$$\iiint_{R^*} f \circ T(u,v,w) \left| \det DT(u,v,w) \right| du\,dv\,dw = \iiint_R f(x,y,z)\,dx\,dy\,dz$$

<u>Ex</u>: $T(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$

$$DT = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & +r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

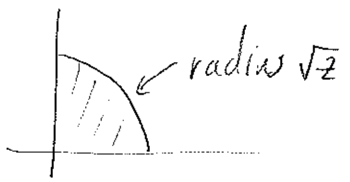
Example from last time:



$$\int_0^2 \int_0^{\sqrt{z}} \int_0^{\sqrt{z-y^2}} x^2+y \, dx\, dy\, dz$$

$$= \int_0^2 \int_0^{\sqrt{z}} \int_0^{\pi/2} \left(r^2\cos^2\theta + r\sin\theta\right) r\, d\theta\, dr\, dz$$

$f(x,y,z) = x^2+y$



$$= \frac{\left(40\pi + 128\sqrt{2}\right)}{240}$$

What about spherical coordinates?

$$DT = \begin{pmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ & \text{Mess} & \\ & & \end{pmatrix}$$

$$\det DT = \rho^2 \sin\phi$$

Ex: Volume of unit sphere $= \int_0^1 \int_0^\pi \int_0^{2\pi} \rho^2 \sin\phi \, d\theta\, d\phi\, d\rho$

$$= \int_0^1 \int_0^\pi 2\pi\rho^2 \sin\phi \, d\phi\, d\rho = \int_0^1 -2\pi\rho^2\cos\phi \Big|_{\rho=0}^{\rho=\pi} d\rho =$$

$$= \int_0^1 4\pi \rho^2 \, d\rho = \frac{4\pi}{3} \rho^3 \Big|_{\rho=0}^{\rho=1} = \frac{4}{3}\pi.$$

But why is it $\rho^2 \sin \phi$?

If time remains give geom reason