Lecture 34: Triple integrals (§6.5)

HW: Due Tuesday April 1 §6.4 7, 13, 18, 29, 30
§6.5 3, 5, 11

Next time: More on §6.5.

Office Hours: None today, Wed @ 4:00-5:30 instead

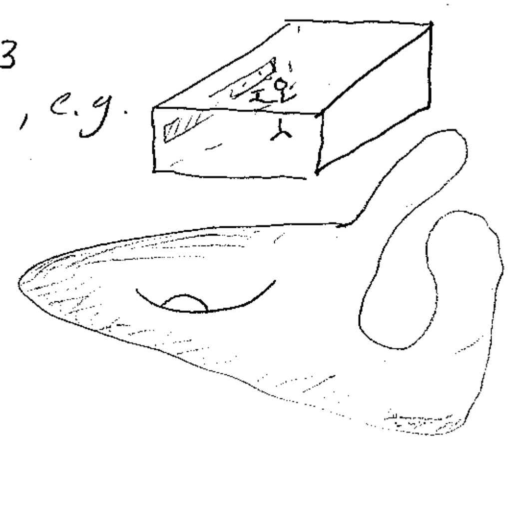
Double integrals:  Region in $\mathbb{R}^2$

$f: \mathbb{R}^2 \to \mathbb{R}$

$$\iint_R f \underbrace{dx\,dy}_{dA} \longrightarrow \iint_R 1\, dx\,dy = \text{Area of } R$$

$$\longrightarrow \text{Average} = \frac{1}{\text{Area}} \iint f\, dA$$

[Computing total mass, etc.]

Triple integrals: R region in $\mathbb{R}^3$, e.g. 

$f: \mathbb{R}^3 \to \mathbb{R}$ some density

$$\iiint_R f(x,y,z)\; dV$$

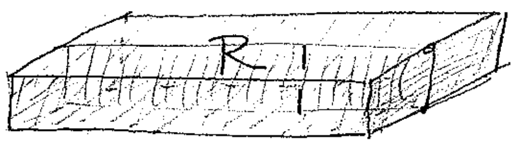
$$\iiint 1\, dV = \text{Volume} \qquad \text{Average} = \frac{1}{\text{Vol}} \iiint f\, dV$$

For instance $\iiint_R 1 \, dV = \text{Volume of } R$

$dV = dx\,dy\,dz$

$$\text{Average of } f \text{ on } R = \frac{1}{\text{Volume of } R} \iiint_R f \, dV.$$

Ex 1:



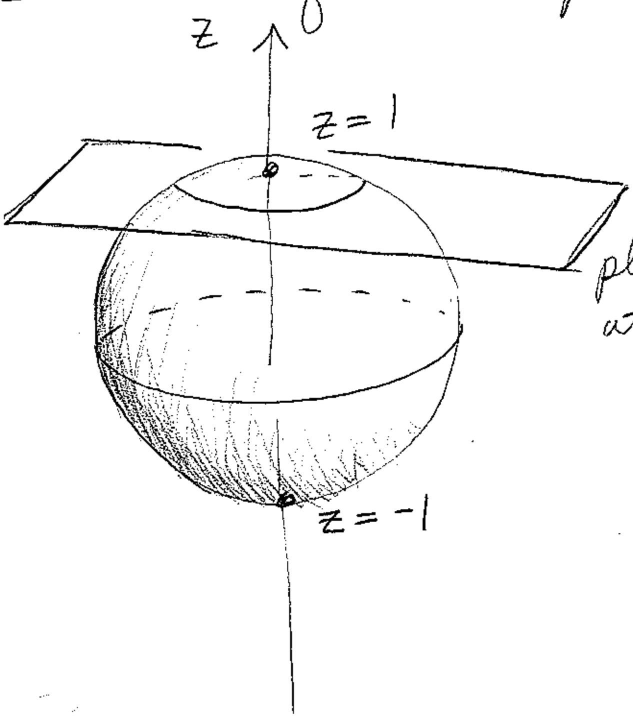
$f(x,y,z) = xy + z$

$0 \leq x \leq 3$
$0 \leq y \leq 2$
$0 \leq z \leq 1$

$$\iiint_R f \, dV = \int_0^2 \int_0^3 \int_0^1 xy + z \, dz \, dx \, dy$$

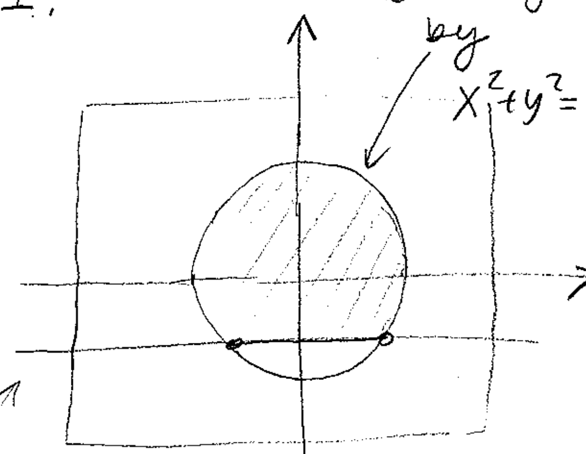
$$= 12$$

Ex 2: Volume of a unit sphere $x^2+y^2+z^2=1$.



plane at height z

circle given by $x^2+y^2=1-z^2$



line at fixed y-height

y runs from $-\sqrt{1-z^2}$
to $+\sqrt{1-z^2}$

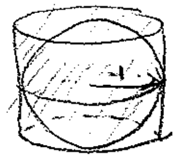
x from $-\sqrt{1-y^2-z^2}$ to $\sqrt{1-y^2-z^2}$

$$\iiint_R 1\, dV = \int_{-1}^{1} \underbrace{\int_{-\sqrt{1-z^2}}^{\sqrt{1+z^2}} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} 1 \, dx\, dy}\, dz$$

Can do with trig substitution or just note that this the area of the circle of radius $\sqrt{1-z^2}$.

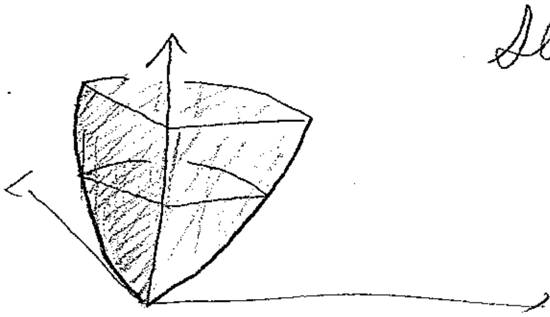
$$= \int_{-1}^{1} \pi (1 - z^2)\, dz = \pi \left(z - \frac{z^3}{3} \right) \Big|_{z=-1}^{z=1}$$

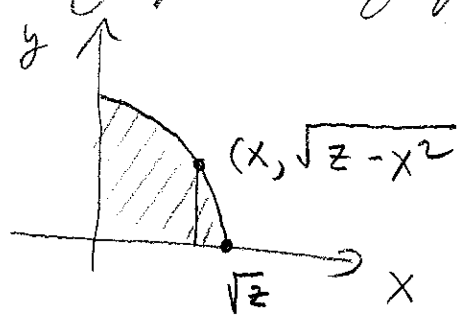
$$= \pi \left(1 - \tfrac{1}{3} - \left(-1 + 1/3 \right) \right) = \pi \left(2 - \tfrac{2}{3} \right) = \frac{4}{3} \pi .$$

Cf. Archimedes.  $\leftarrow$ area 2π so $\dfrac{\mathrm{Vol}(\ )}{\mathrm{Vol}\ } = \dfrac{3}{2}$.

<u>Ex</u>^{3}: Consider the region in the pos. octant bounded by the planes $x = 0,\ y = 0,\ z = 2$ and the surface $z = x^2 + y^2$

Slice by planes of fixed z-height



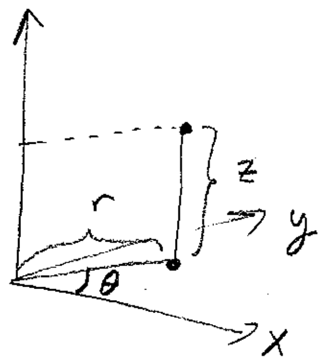


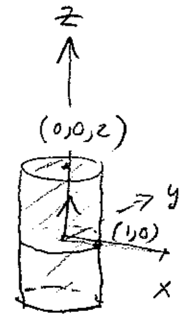
$$\iiint_R f\,dV = \int_0^2 \int_0^{\sqrt{z}} \int_0^{\sqrt{z-x^2}} f(x,y,z)\,dy\,dx\,dz$$

Cylindrical Coordinates:

$(r, \theta, z) \longleftrightarrow (r\cos\theta,\ r\sin\theta,\ z)$

$0 \le r$
$0 \le \theta \le 2\pi$





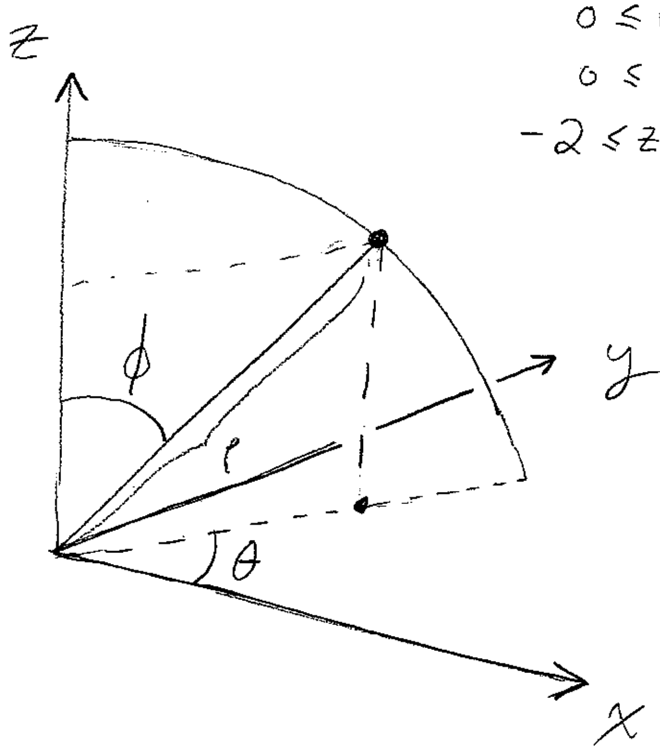
$0 \le r \le 1$
$0 \le \theta \le 2\pi$
$-2 \le z \le 2$

Spherical Coordinates

$0 \le \rho$

$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$



$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$0 \le \rho \le 1$
$0 \le \phi \le \pi$
$0 \le \theta \le 2\pi$