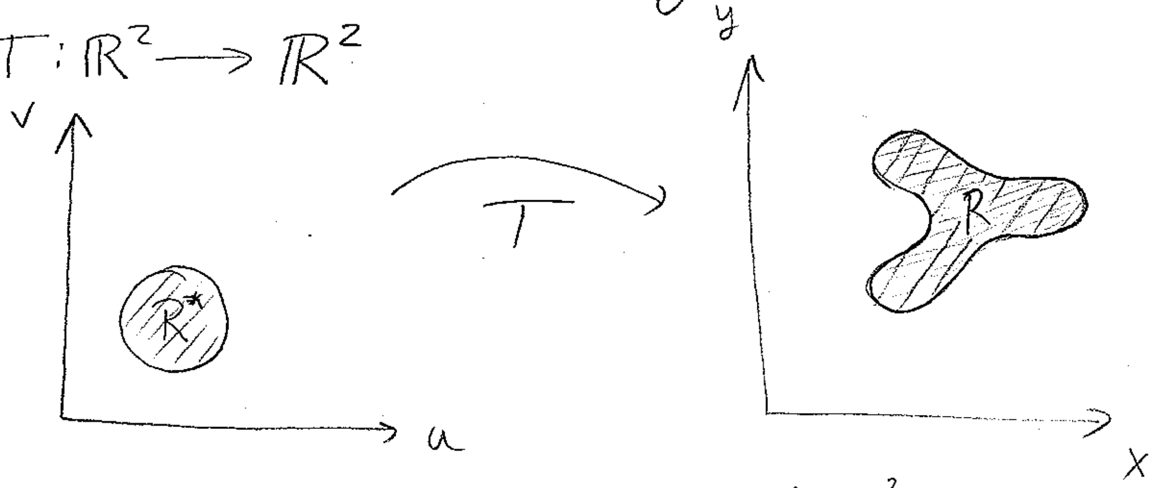
Lecture 33 More on change of variables

$T : \mathbb{R}^2 \to \mathbb{R}^2$

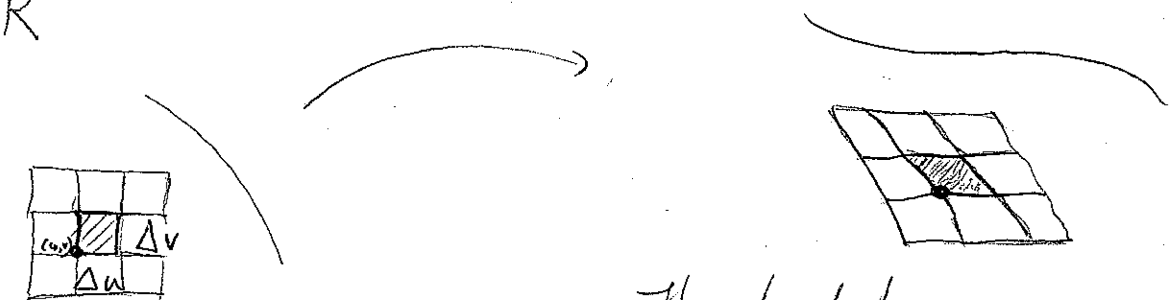


$T(R^*) = R.$ $f : \mathbb{R}^2 \to \mathbb{R}$

Provided T is reasonably 1-1 then

$$\iint_{R^*} f \circ T(u,v) \, |\det DT(u,v)| \, du \, dv = \iint_R f(x,y) \, dx \, dy$$

Reason:



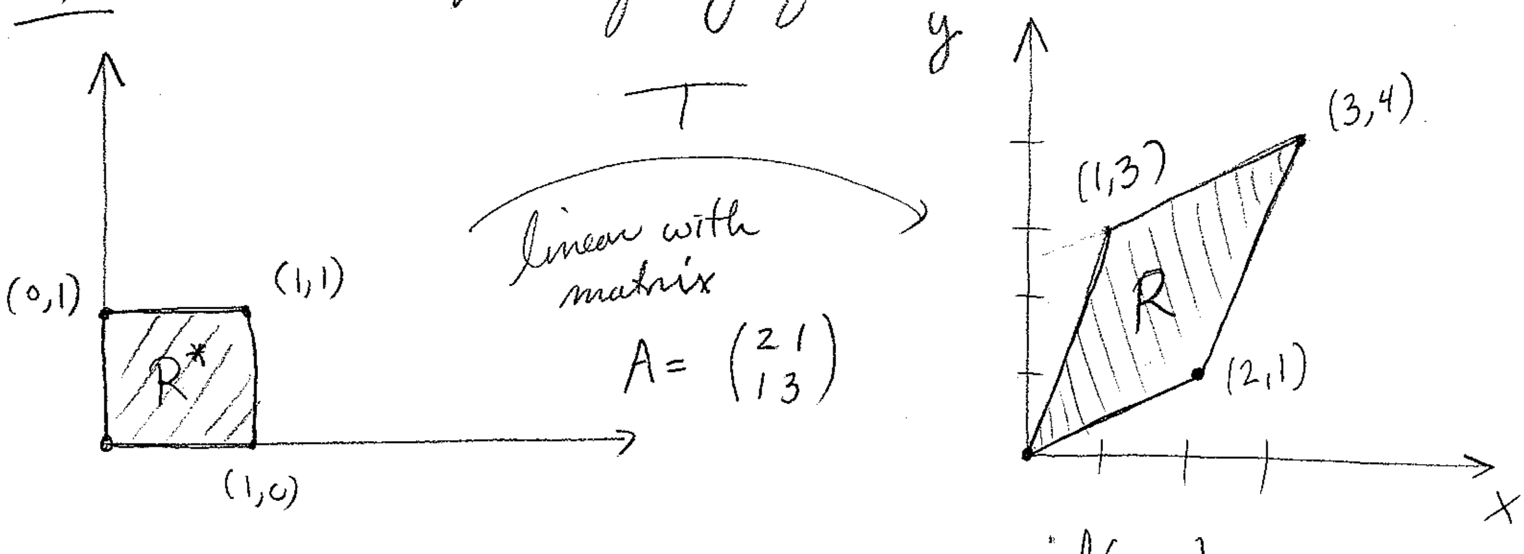
The shaded square contributes to the Riemann sum the ammount

$f(\bullet) \cdot \text{Area} \simeq$

$$f(T(u,v)) \cdot |\det DT(u,v)| \Delta u \Delta v$$

Reasonably 1-1 means there are few points where $T(\vec{X}_0) = T(\vec{X}_1)$ and $\vec{X}_0 \neq \vec{X}_1$. Cf. polar coordinates.

Ex: [Start on right, query for the matrix T.]



$f(x,y) = xy$

$$T(u,v) = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u+v \\ u+3v \end{pmatrix}$$

$DT = A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ indep of $u,v \Rightarrow |\det DT| = 5$

So

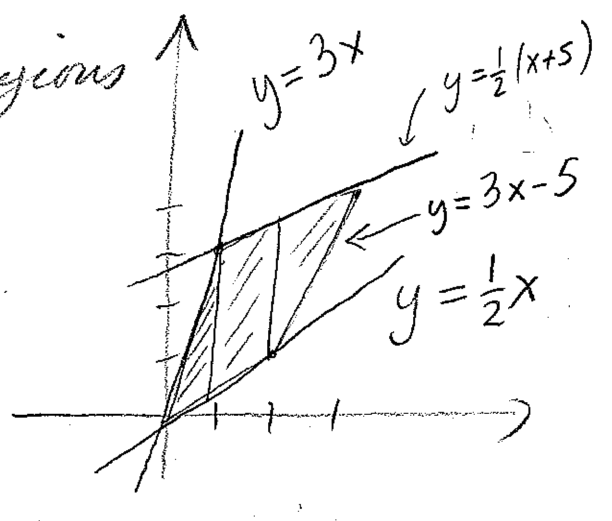
$$\iint_R xy\,dx\,dy = \iint_{R^*} \underbrace{(2u+v)(u+3v)}_{f\circ T(u,v)} \underbrace{5}_{|\det DT|}\,du\,dv$$

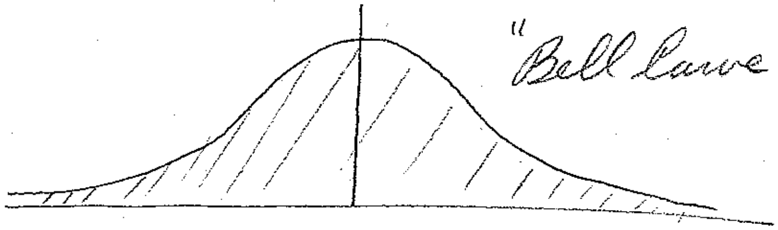
$$= 5\int_0^1\int_0^1 2u^2 + 7uv + 3v^2\,du\,dv = \frac{205}{12}$$

Compare approach using elementary regions where the integrals over the regions shown are

$\frac{35}{32}$, $\frac{365}{48}$, $\frac{805}{96}$

respectively.



Ex: e^{-x^2}  "Bell Curve"

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = m \quad \left[\text{can't do in elementary terms}\right]$$

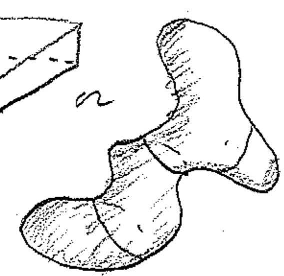
$$m^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{e^{-x^2} e^{-y^2}}_{e^{-(x^2+y^2)}}\,dx\,dy$$

$$= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

$u = -r^2$

$du = -2r\,dr$

$$= \int_0^{2\pi} -\frac{1}{2} e^{-r^2}\Big|_{r=0}^{r=\infty}\,d\theta = \pi. \quad \text{Thus } \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Tripple Integrals: R a region in $\mathbb{R}^3$, e.g.  or 

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}$, can discuss

$$\iiint_R f(x,y,z)\,dx\,dy\,dz$$