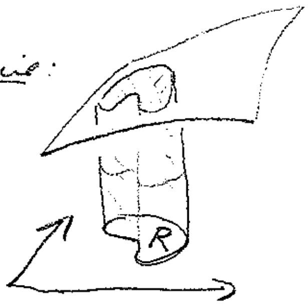
Lecture 32: Change of variables (§6.4)

HW: On web, due Thurs. OH: Today 4-5:30.

Next time: More on §6.4.

Last time:



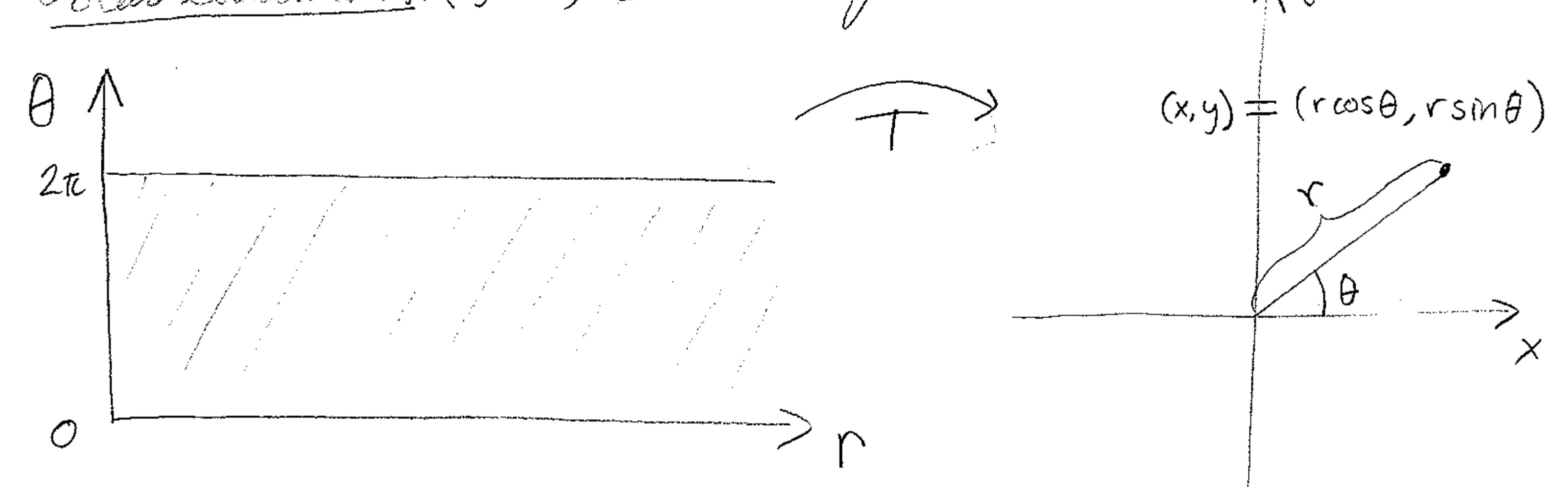
$\iint_R f\, dA$

Area of 

$= \iint_R 1\, dA$

[Change of vars is one way we can handle complicated regions.]

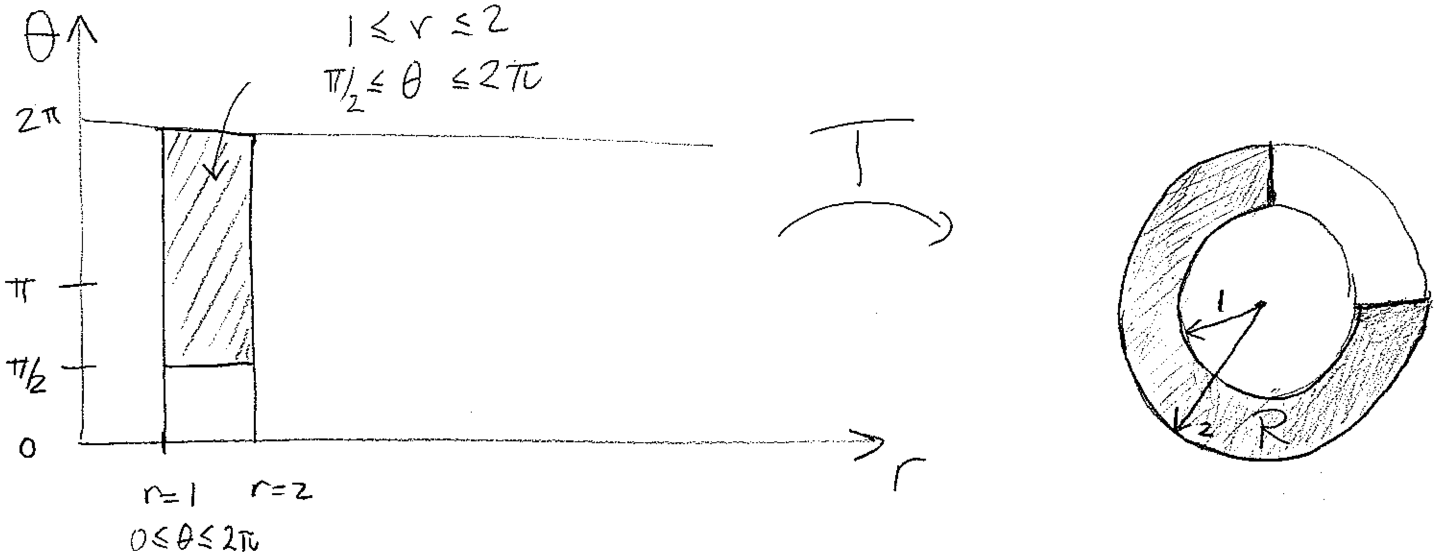
Polar Coordinates: (§2.8) [Start w/ right side]



$0 \leq r \qquad 0 \leq \theta \leq 2\pi$

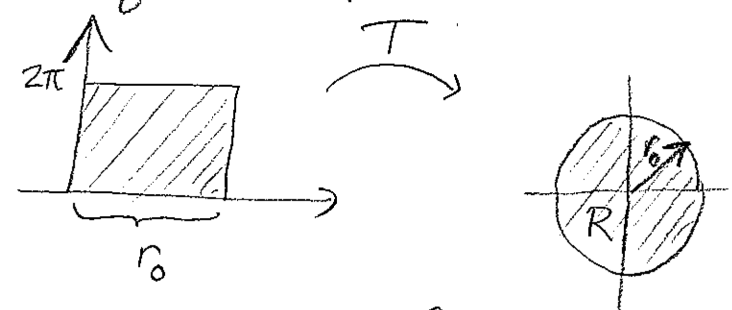
where $T(r,\theta) = (r\cos\theta, r\sin\theta)$

[Point: Some object are easier to describe in polar coor]



What happens if we try to do integrals in polar coord.

R = disc of radius r_0



$$\int_0^{2\pi}\int_0^{r_0} 1 \, dr \, d\theta = 2\pi r_0 \neq \text{Area of } R = \pi r_0^2 = \iint_R 1 \, dx\,dy$$

However, we can fix this. The right thing is

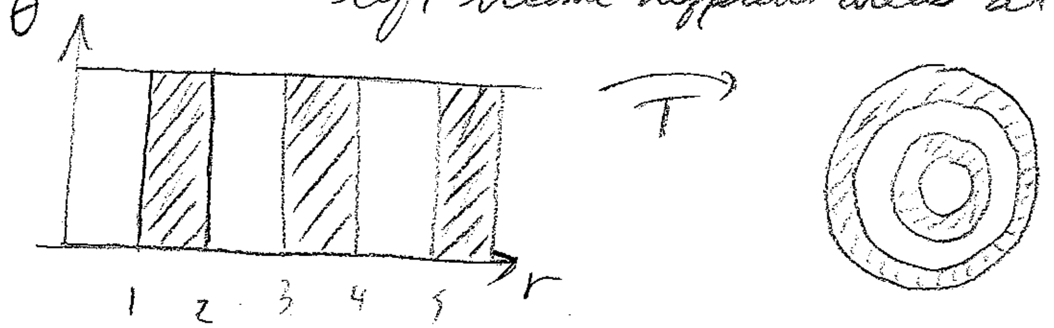
$$dx\,dy \longleftrightarrow r\,dr\,d\theta$$

For instance

$$\int_0^{2\pi}\int_0^{r_0} 1 \, r \, dr \, d\theta = \int_0^{2\pi} \frac{r^2}{2}\Bigg|_{r=0}^{r_0} d\theta = \pi r_0^2 = \text{Area!}$$

[Rest of this lecture is devoted to understanding why $dx\,dy$ "=" $r\,dr\,d\theta$ and how to find such expressions for arb. coordinates.]

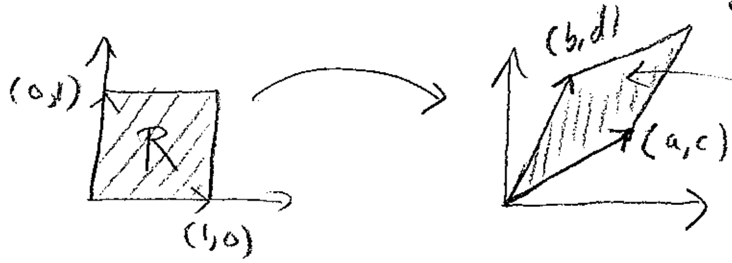
Problem: T distorts area: equal area regions at left become different areas at right



[The "r" in $r\,dr\,d\theta$ accounts for this.]

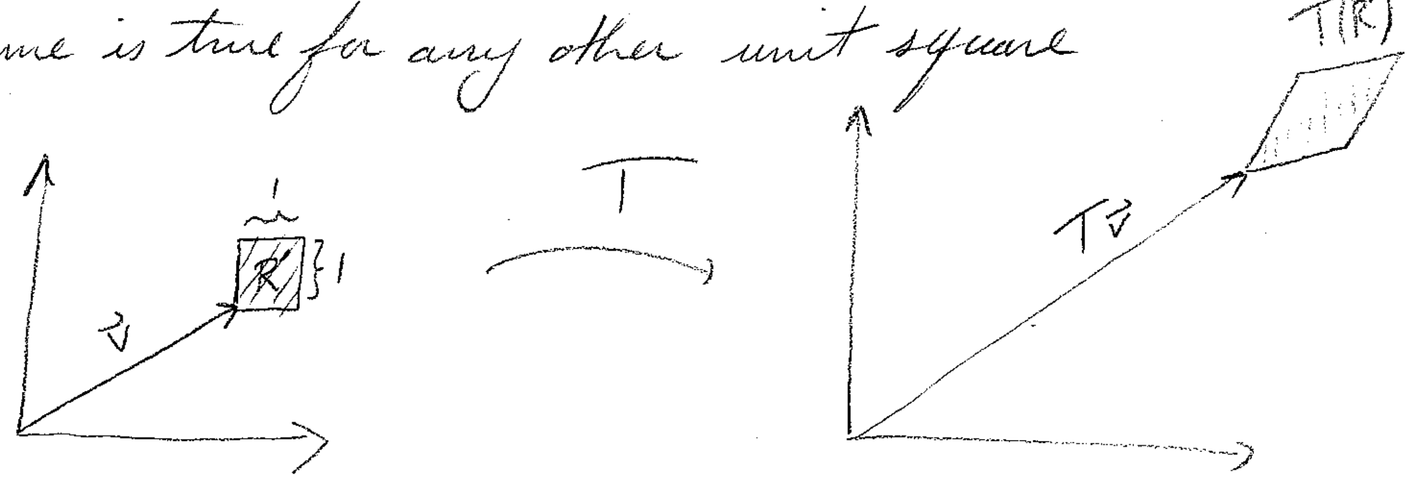
Q: How does a function $T: \mathbb{R}^2 \to \mathbb{R}^2$ distort area?

Start with a linear trans T given by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$



has area $= |\det A|$

Same is true for any other unit square



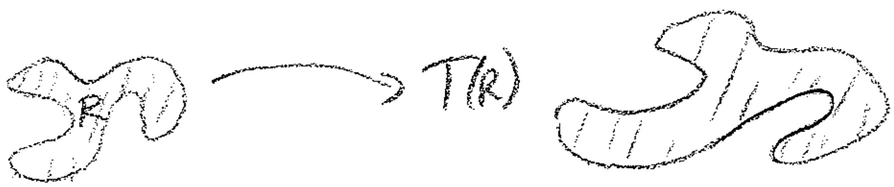
since it is taken to a translate of the image of the standard one.

The same happens to small squares

Area $1/4$ $\longrightarrow$ Area $\frac{1}{4}|\det A|$

<u>Conclusion</u>: If R is <u>any</u> region, then

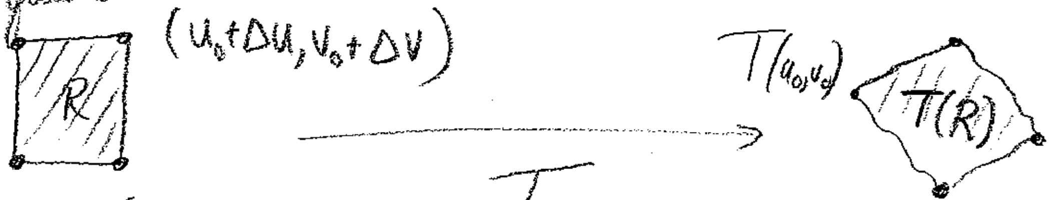
$$\text{Area}(T(R)) = |\det A| \, \text{Area}(R)$$



Now consider a general [differentiable] function

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(u,v) \qquad (x,y)$$

Small square:

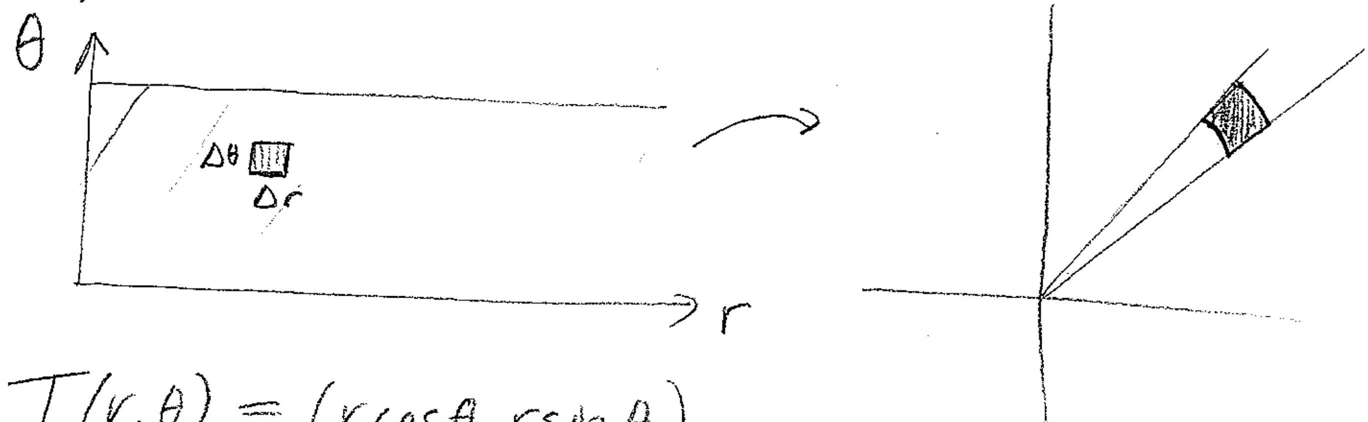


If T is differentiable at (u_0,v_0) then $T(R)$ is approx. the parallelogram gotten by applying $DT(u_0,v_0)$ to

since $T(u_0+\Delta u, v_0) = T(u_0,v_0) + DT\begin{pmatrix}\Delta u\\ 0\end{pmatrix}$ etc.

Thus, the image of R has area $\approx |\det DT(u_0,v_0)|\,\Delta u \Delta v$.

For polar coor



$T(r,\theta) = (r\cos\theta, r\sin\theta)$

$$DT = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \quad \text{so } \det(DT) = r$$

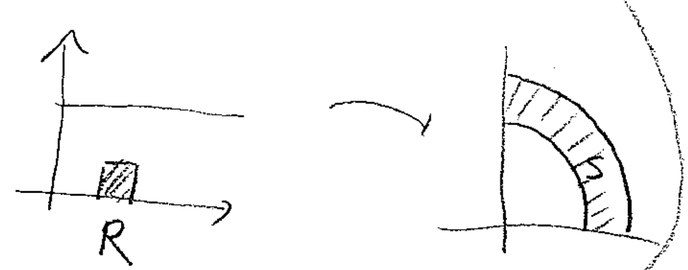
and image of $\square\,\Delta\theta$, Δr has area $\approx \underbrace{r\,\Delta r\,\Delta\theta}$.

connect!

Thus, to compute the area of a region S in polar coor, find R with $T(R)=S$

Then

$$\iint_S 1\,dx\,dy = \iint_R \underbrace{r\,dr\,d\theta}$$



Works for any function

Ex: $\iint_S \sin(x^2+y^2)\,dx\,dy$

$$= \int_0^{\pi/2}\int_1^2 \sin\left((r\cos\theta)^2 + (r\sin\theta)^2 \right) r\,dr\,d\theta$$

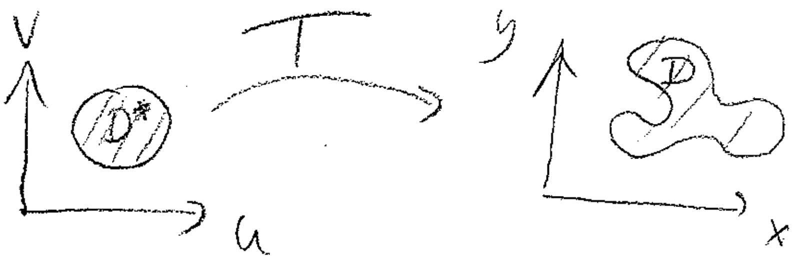
$$= \int_0^{\pi/2}\int_1^2 \sin(r^2)\, r\,dr\,d\theta = \int_0^{\pi/2}\int_1^4 \frac{1}{2}\sin u\,du\,d\theta$$

$u = r^2$

$du = 2r$

$$= \int_0^{2\pi} -\frac{1}{2}(\cos 4 - \cos 1)\,d\theta = -\pi(\cos 4 - \cos 1)$$

In general



$T(D^*) = D$

If T is "reasonable", then

$$\iint_D f\,dx\,dy = \iint_{D^*} f(T(u,v))\,|\det DT(u,v)|\,du\,dv.$$