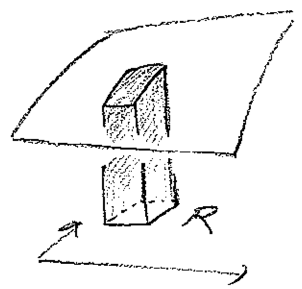
Lecture 31: More on multivariable integration

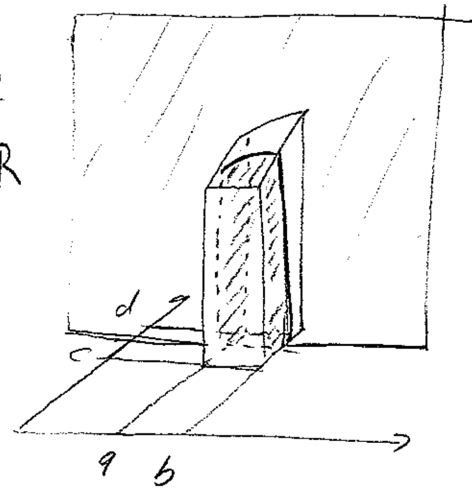
HW: Web. Next time:

Last time:

Volume = $\iint_R f(x,y)\,dx\,dy$



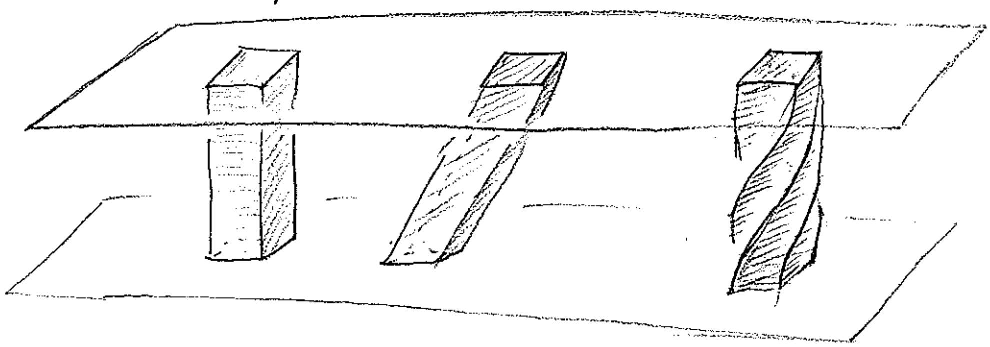
graph of $f:\mathbb{R}^2\to\mathbb{R}$



Slice over line a height y has area $A(y)$

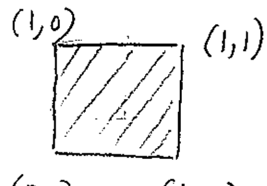
$$\iint_R f(x,y)\,dx\,dy = \int_c^d A(y)\,dy = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$$

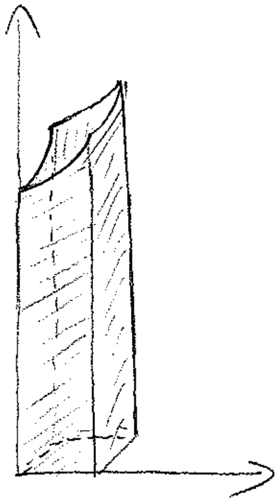
Cavalieri's Principle:



All have equal volume!

Ex: $f(x,y) = x^2+y^2+5 \qquad R =$



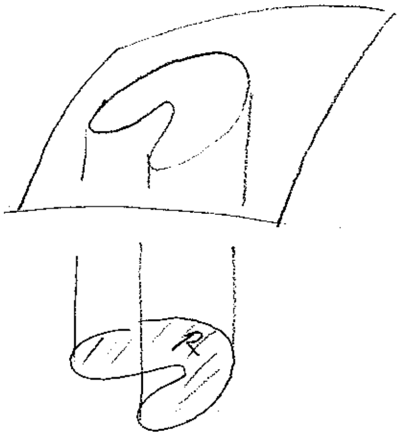


$$\iint_R f(x,y)\,dx\,dy = \int_0^1\left(\int_0^1 x^2+y^2+5\,dx\right)dy$$

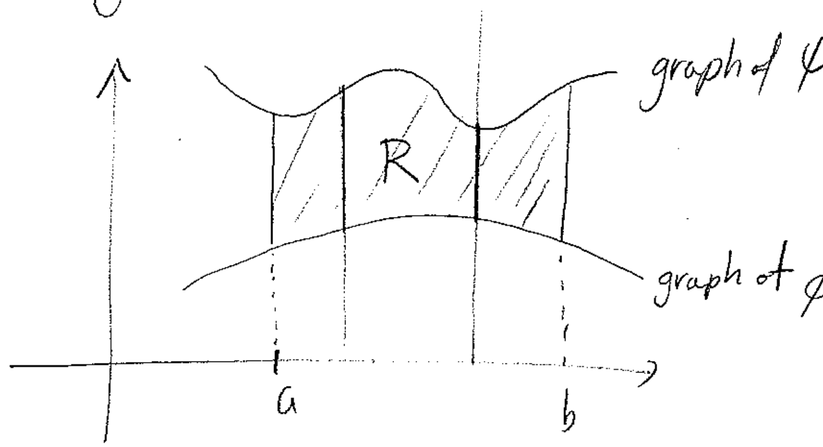
$$= \int_0^1 \frac{x^3}{3}+(y^2+5)x\Big|_{x=0}^{x=1} dy = \int_0^1 y^2+5+1/3\,dy$$

$$= \frac{y^3}{3} + 16/3\, y\Big|_{y=0}^{1} = 17/3 = 5\,2/3$$

Regions R that are not rectangles:



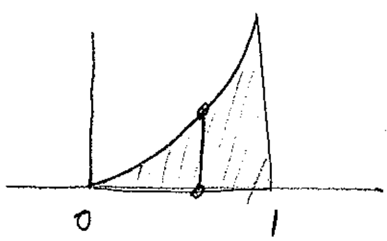
Elementary region (Type 1)



$$R = \{ a \le x \le b \text{ and } \phi(a) \le x \le \psi(b) \}$$

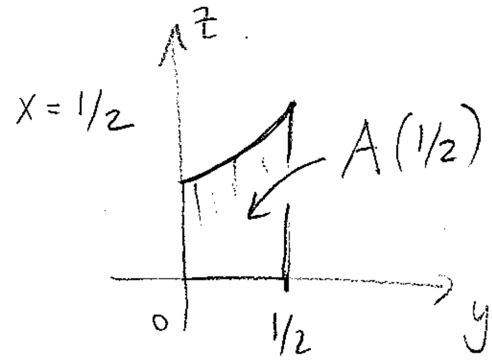
Point: Slice along lines $x = c$

Ex: $R =$



$\psi(x) = x^2$

$\phi(x) = 0$

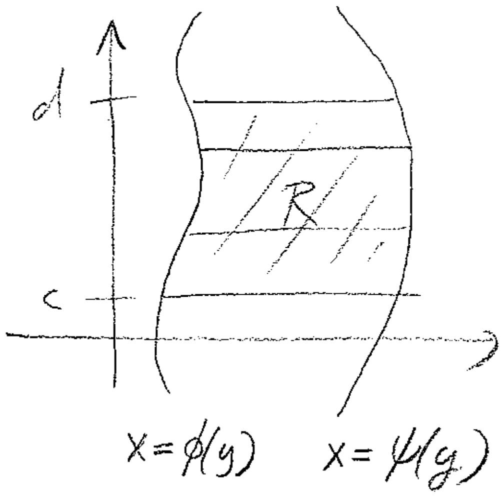


$$\iint_R xy + y^2 \, dx\,dy = \int_0^1 A(x)\, dx = \int_0^1 \left(\int_0^{x^2} xy + y^2 \, dy \right) dx$$

$$= \int_0^1 \tfrac{1}{2} x y^2 + y^3/3 \Big|_{y=0}^{y=x^2} dx = \int_0^1 \tfrac{1}{2} x^5 + \tfrac{1}{3} x^6 \, dx$$

$$= \tfrac{1}{12} x^6 + \tfrac{1}{21} x^7 \Big|_{x=0}^{1} = \tfrac{11}{84}.$$

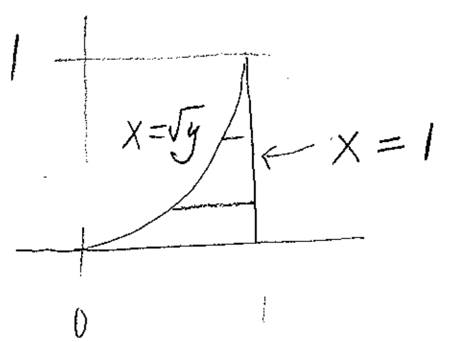
Type 2:



R: $c \le y \le d$

$\phi(y) \le x \le \psi(y)$

Return to previous region [which is "type 3".]



$$\iint_R xy+y^2 \, dx\,dy =$$

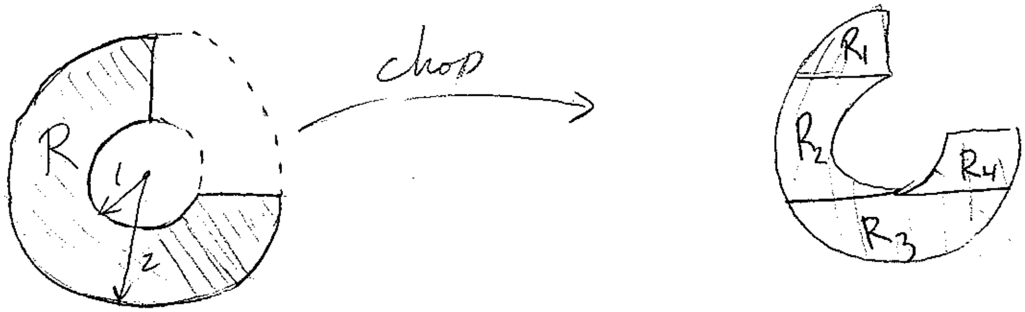
$$= \int_0^1 \left(\int_{\sqrt{y}}^1 xy+y^2 \, dx \right) dy = \int_0^1 \tfrac{1}{2}x^2y + y^2x \Big|_{x=\sqrt{y}}^{1} dy$$

$$= \int_0^1 \tfrac{1}{2}y + y^2 - \tfrac{1}{2}y^2 - y^{5/2} \, dy = \int_0^1 \tfrac{1}{2}y + \tfrac{1}{2}y^2 - y^{5/2} \, dy$$

$$= \tfrac{1}{4}y^2 + \tfrac{1}{6}y^3 - \tfrac{2}{7}y^{7/2} \Big|_{y=0}^{1} = \tfrac{5}{12} - \tfrac{2}{7} = \tfrac{11}{84}. \checkmark$$

This is the same as before since they both compute the same volume. [Sometimes, one way is much easier than another ...]

What about more complicated regions?

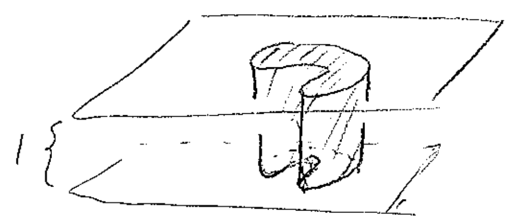


$$\iint_R f\,dx\,dy = \iint_{R_1} f\,dx\,dy + \iint_{R_2} f\,dx\,dy + \iint_{R_3} f\,dx\,dy + \iint_{R_4} f\,dx\,dy$$

[Another method is changing coordinates, will talk about next time.]

<u>Other interpretations:</u> 1)  in $\mathbb{R}^2$

$$\iint_R 1\,dx\,dy = \text{Area of } R$$



For this reason, sometimes write $\iint_R f\,dA$ for $\iint_R f\,dx\,dy$.

2) Averages; $f: \mathbb{R}^2 \to \mathbb{R}$

$$\begin{matrix}\text{Average of} \\ f \text{ on } R\end{matrix} = \frac{1}{\text{Area of } R} \iint_R f\,dA$$

3) R made of material of density given by f

total mass $= \iint_R f\,dA$.

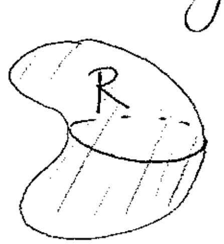
If time remains:

1) C a curve param by $c:[a,b]\to\mathbb{R}^3$, $\vec{F}$ a vector field (F_1,F_2,F_3)

$$\int_C \vec{F}\cdot ds = \int_C F_1\,dx + F_2\,dy + F_3\,dz$$

$$dx = c_1'(t)\,dt$$

2) 3-dim'l integration



in $\mathbb{R}^3$

$$\iiint_R 1\,\underset{dxdydz}{\underset{\parallel}{dV}} = \text{Volume}$$

$$\frac{1}{\text{Volume}}\iiint_R f\,dV = \text{average of } f \text{ on } R\text{, etc.}$$