Lecture 30: More on curl; Intro to multivar. integration.

HW: Handout. Return exams.

Next time: §6.1–6.2

Earlier in Math 241: $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ a vector field

$$\vec{F} = (F_1, F_2, F_3)$$

An associated vector field is

$$\operatorname{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) i - \cdots$$

$\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ a vector field given by $(F_1(x,y), F_2(x,y))$

If we "promote" to $\underline{F} = (F_1(x,y), F_2(x,y), 0)$ on $\mathbb{R}^3$, then

$$\operatorname{curl} \underline{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & 0 \end{vmatrix} = \underbrace{\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)}_{\text{scalar curl}} k$$

Suppose $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ is conservative, $F = \nabla f$, $f: \mathbb{R}^2 \to \mathbb{R}$

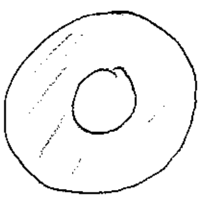
Then

$$\text{s.curl } F = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial f}{\partial x \partial y} - \frac{\partial f}{\partial y \partial x} = 0.$$

Ex: $\vec{F} = (y, 0)$ not conservative since path dependent, or since s. curl $\vec{F} = 1$.

Q: $\vec{F}$ a vector field defined on some region U of $\mathbb{R}^2$. If s. curl $\vec{F} = 0$, must F be conservative?

A. Yes if $U = \mathbb{R}^2$ or if U "has no holes" [simply connected] [Will talk about this later in Chapter 8.] But not always



$U = \{ 1 < \|\vec{x}\| < 2 \}$ $\quad F(x,y) = \dfrac{1}{x^2+y^2}(-y, x)$

has s. curl $= 0$ but is not conservative. [HW.]

[Similar story in $\mathbb{R}^3$ with curl$(\nabla f) = \vec{0}$.]

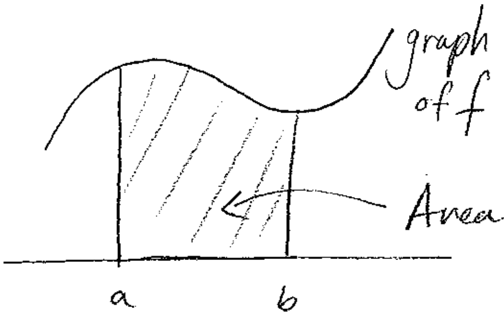
Notation: C a curve param by $c: [a,b] \to \mathbb{R}^3$, $\vec{F}$ a vector field
$c = (c_1, c_2, c_3)$ $\quad (F_1, F_2, F_3)$

$\int_C \vec{F} \cdot ds$ is sometimes written

$$\int_C F_1\,dx + F_2\,dy + F_3\,dz, \quad dx = c_1'(t)\,dt \text{ etc.}$$

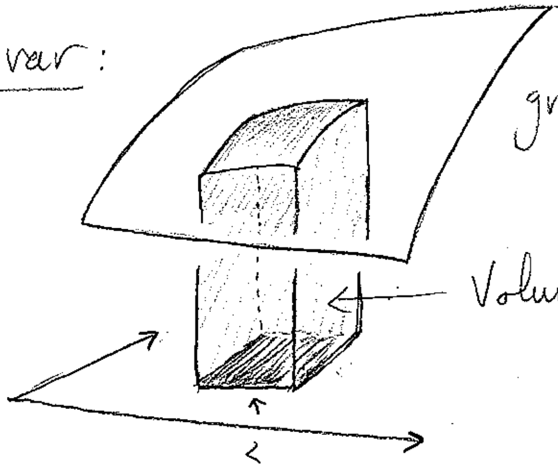
[As mentioned will repeat the curve picture for surfaces in $\mathbb{R}^3$, but first we need learn how to do multi-dimensional integrals...]

Onevar:



$\text{Area} = \int_a^b f(x)\,dx$ [Computed using fund. thm. of calc.]

Two var:



graph of $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

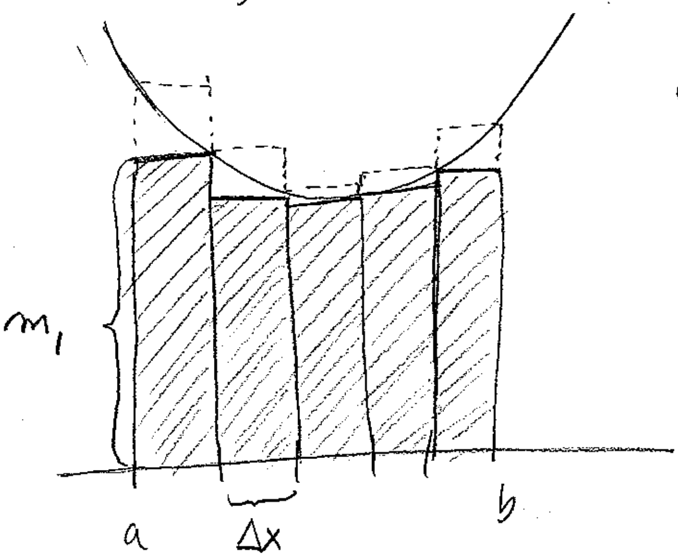
Volume is $\iint_R f(x,y)\,dx\,dy$

Base a square R

Q1: What does this mean mathematically?

Q2: How do we compute it?

[In one var, we addressed the first question as.]



on i^{th} small interval f has min m_i and max M_i

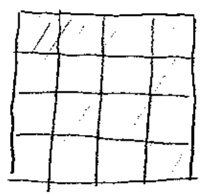
Thus

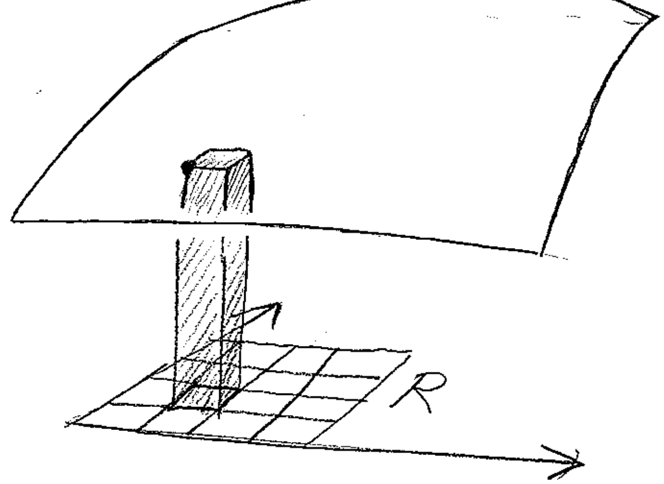
$$\sum_{i=1}^{n} m_i \Delta x \leq \int_a^b f(x)\,dx \leq \sum_{i=1}^{n} M_i \Delta x$$

As $\Delta x \to 0$ these two bounds converge to the middle.

Divide $[a,b]$ into segments of length Δx.

In two vars:

R: 



Each square $\underbrace{\square}_{\Delta x}\}\Delta y$ has area $\Delta x \Delta y$.

Box shown has height = min of f on subsquare.

Thus

$$\sum_{\substack{\text{small} \\ \text{squares}}} \left(\begin{matrix}\text{min value of } f \\ \text{on subsquare}\end{matrix}\right) \Delta x \Delta y \le \iint_R f(x,y)\,dx\,dy \le$$

$$\sum_{\substack{\text{small} \\ \text{squares}}} \left(\begin{matrix}\text{max of } f \\ \text{on subsquares}\end{matrix}\right) \Delta x \Delta y$$

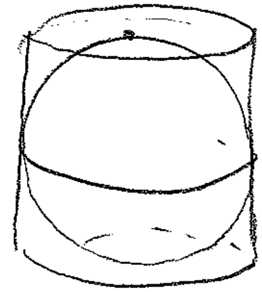
As $\Delta x, \Delta y \to 0$, then [provided f is continuous] these two bounds converge to define the integral.

OK, but how do we compute?

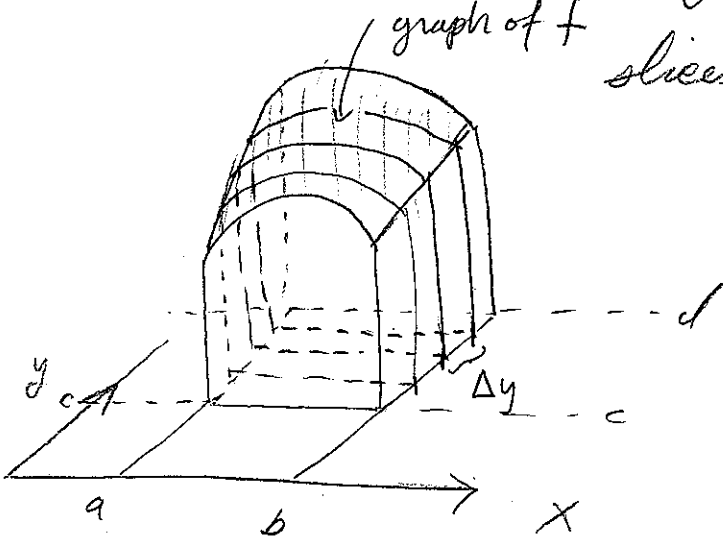
<u>Archimedes:</u> (225 B.C.E)

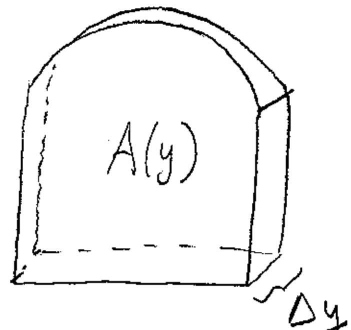
3 : 2

volume + surface area!



Can reduce to one var integrals by cutting into slices





which have volume $\approx A(y)\Delta y$

where $A(y)$ is the area of front of the slice.

To get the total volume, add vols of slices

$$\iint_R f(x,y)\,dx\,dy = \int_c^d A(y)\,dy = \int_c^d \Big(\underbrace{\int_a^b f(x,y)\,dx}_{y \text{ fixed}}\Big)\,dy$$

$$= \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx$$