Lecture 11: Limits and continuity.

Last time: $E : \mathbb{R}^2 \to \mathbb{R}$. Then $\lim_{\vec{h} \to \vec{0}} E(\vec{h}) = 0$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that when $\|\vec{h}\| < \delta$ then $|E(\vec{h})| < \varepsilon$.

HW: Handout

Next time: Derivatives (§2.4)

$$f(x,y) = \frac{xy^2}{x^2+y^4}$$

for $(x,y) \neq (0,0)$

What does f do near $\vec{0}$?

Along the line $y = cx$

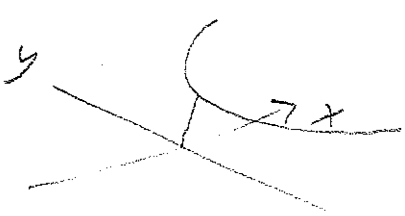
$$f(x, cx) = \frac{c^2x^3}{x^2 + c^4x^4} = \frac{c^2x}{1+c^4x^2}$$

which goes to 0 as $x \to 0$.

But: Consider the parabola $x = y^2$

along this f is

$$f(y^2, y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = 1/2$$



So the limit does not exist.

[Illustrates the flaws of just looking a slices]

Rules for limits: $f, g : \mathbb{R}^n \to \mathbb{R}^m$

$$\lim_{\vec{x}\to\vec{a}} \left(f(\vec{x}) + g(\vec{x})\right) = \lim_{\vec{x}\to\vec{a}} f(\vec{x}) + \lim_{\vec{x}\to\vec{a}} g(\vec{x})$$ [Provided that these make sense]

[What about a product rule? Since it only makes sense to mult vectors under very special circumstances, consider...]

$f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ then

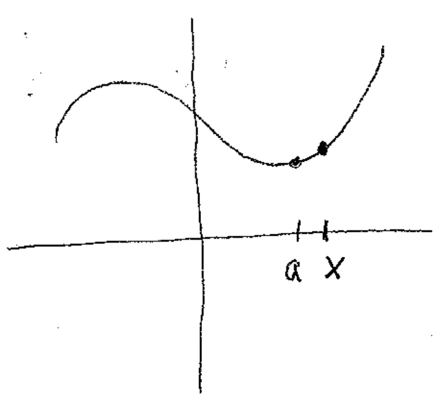
$$\lim_{\vec{x}\to\vec{a}} f(\vec{x})\, g(\vec{x}) = \left(\lim_{\vec{x}\to\vec{a}} f(\vec{x})\right)\left(\lim_{\vec{x}\to\vec{a}} g(\vec{x})\right)$$

$$\lim_{\vec{x}\to\vec{a}} \frac{1}{f(\vec{x})} = \left(\lim_{\vec{x}\to\vec{a}} f(\vec{x})\right)^{-1}$$

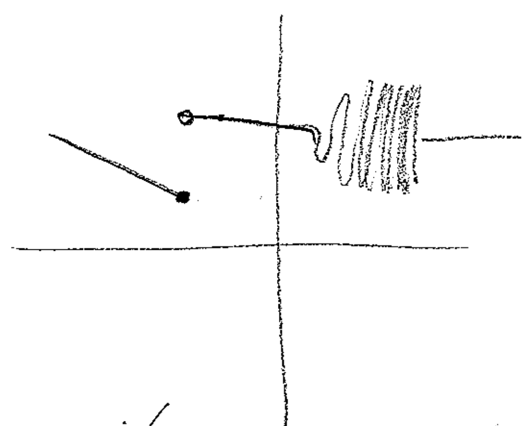
provided this is $\neq 0$.

[True for ess. the same reasons as one variable.]

Continuity:



vs.



A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if

for each $\vec{a}$ we have $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$

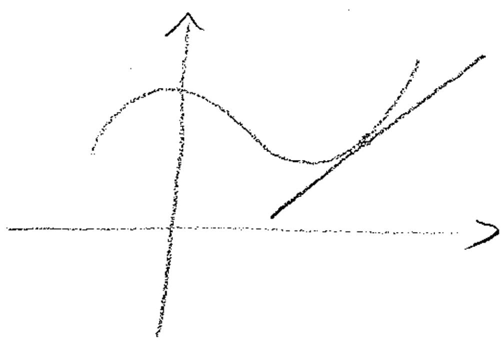
Ex: $f(x,y) = \begin{cases} \dfrac{x^2}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

is continuous. For $(x,y) \neq (0,0)$ this is because it is "built up" out of continuous pieces. [Only tricky part is $\sqrt{}$ but composition of continuous functions is continuous.] For $(0,0)$ need to check $\lim\limits_{\vec{x} \to 0} \dfrac{x^2}{\sqrt{x^2+y^2}} = 0$ where $\vec{x} = (x,y)$.

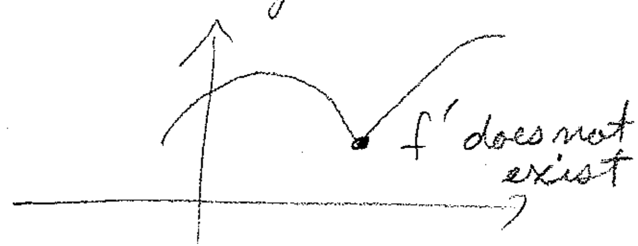
Now $\sqrt{x^2+y^2} = \|\vec{x}\|$ and $x^2 \leq \|\vec{x}\|^2$. So $\left|\dfrac{x^2}{\sqrt{x^2+y^2}}\right| \leq \|\vec{x}\|$

So given ε take $\delta = \varepsilon/2$.

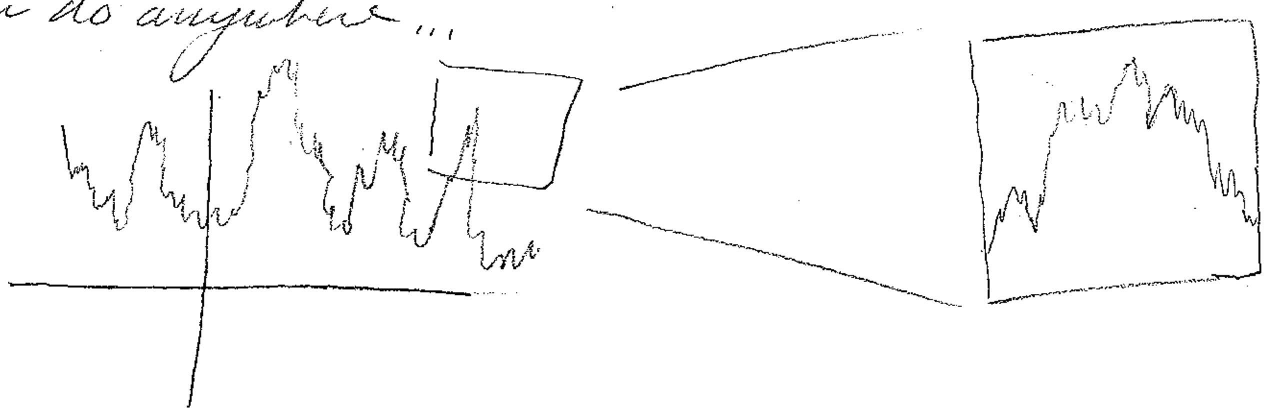
Derivatives: Approximation by a line



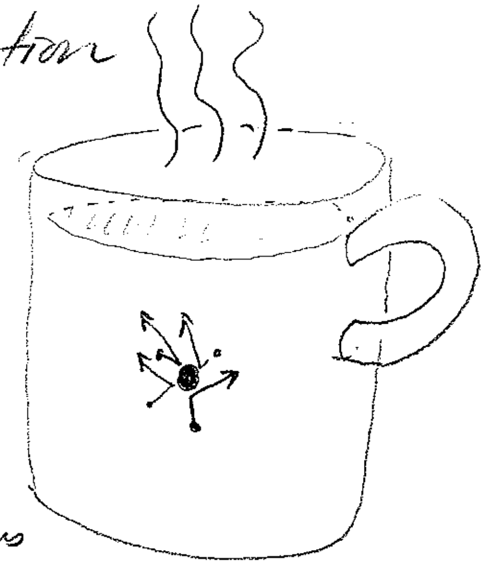
Can't always do



Or even do anywhere...



examples: Stock market ; Brownian motion

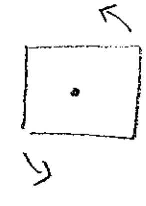


Think dust moving in sunlight coming in through a window. Brown (19^{th} century) was looking at pollen grains moving on the surface of water. Einstein (1905) brought to the attention of physicists. However, 2000 years earlier, the Roman Lucretius used this as an argument for the existence of molecules....

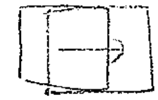
In this class, we're going to look almost exclusively at differentiable functions. There are still some things you can say about a general continuous function.

Suppose $f: \mathbb{R}^n \to \mathbb{R}^n$. A fixed point of f is one where $f(\vec{x}) = \vec{x}$

Ex: $f(x,y) = (-x,-y)$ has fixed point $(0,0)$



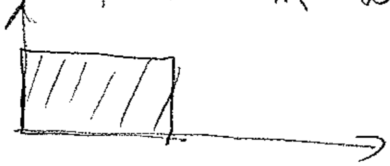
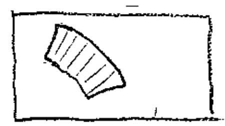
$f(x,y) = (x+1, y)$ has no fixed points



Important in Economics, for example, where market equilibria can often be characterized in this way.

Brower Fixed Point Theorem:

Suppose $f:\mathbb{R}^2 \to \mathbb{R}^2$ which takes a rectangle

$R =$  into itself 

Then f has a fixed point.

Demonstrate with *Daily Illini*.