Lecture 10: Limits of functions of several variables

HW: No additional

Next time: More on limits

Calculus was discovered / invented by Newton and Leibnitz in the late 17^{th} century, in pretty much its current form. (Notation comes mostly from L. N. used terms "fluxion" and "fluent.") Roots go back 2300 years to Archimedes, who computed the area under a parabola  and determined the volume of the sphere. However, ε-δ's weren't introduced until Cauchy in the 1820s. Reasons: 1) Philosophy (Berkeley)

2) consolidation.

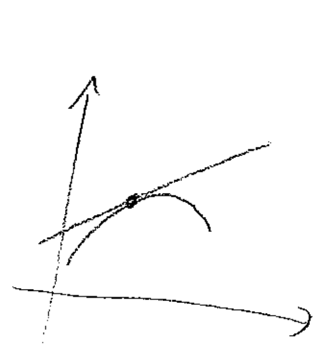
3) teaching.

$\boxed{\text{4) estimates of } \underline{\text{error}}}$ ← point of view I'll take today

[N.B. the Vector Calculus proper was introduced in the 2^{nd} half of the 19^{th} century.]

What does a calculator do we ask it for

$$\sin(2) = 0.9092974268$$



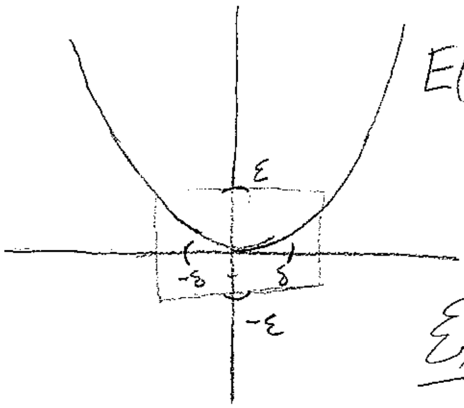
It uses

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Consider $E:\mathbb{R}\to\mathbb{R}$. (An "Error function.") We say

error bound

$\lim\limits_{h\to 0} E(h) = 0$ if given $\boxed{\varepsilon} > 0$ we can always find δ so that whenever $|h| < \delta$ then $|E(h)| < \varepsilon$



$E(h) = h^2$

Can view as a challenge-response process.

Ex: $\varepsilon = 1/10 \qquad \delta = 1/4$

If $|h| < 1/4$ then $|E(h)| = |h|^2 < 1/16 < 1/10 = \varepsilon$

Let's check $\lim\limits_{h\to 0} h^2 = 0$.

If you give me $\varepsilon > 0$, I'll take $\delta = \sqrt{\varepsilon}$. Thus if $|h| < \delta$ then $|h^2| = |h|^2 < \delta^2 = \varepsilon$

Ex: $E(h) = 2h + h^2 \qquad \varepsilon = 1/10 \qquad \delta = 1/100$

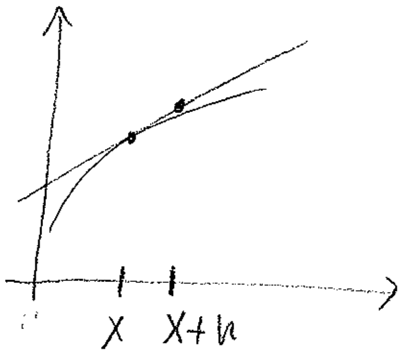
$\lim\limits_{h\to 0} 2h + h^2 = 0$

$$|2h + h^2| \leq 2|h| + |h|^2 = \frac{2}{100} + \frac{1}{10000} < \frac{3}{100} < \varepsilon$$

In general, say

$$E(h) = f(a+h) - c$$

$\lim\limits_{x\to a} f(x) = c$ if $f(a+h) = c + E(h)$ where $\lim\limits_{h\to 0} E(h) = 0$.



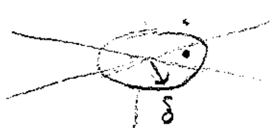
$$f(x+h) = f(x) + f'(x)h + E(h)$$

where $E(h)$ is <u>really</u> small: $\displaystyle\lim_{h\to 0} \frac{E(h)}{h} = 0$.

[Similarly with the other terms in the Taylor expansion.]

Now suppose $E: \mathbb{R}^2 \to \mathbb{R}$. We say

$\displaystyle\lim_{\vec{h}\to 0} E(\vec{h}) = 0$ if given $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $\|\vec{h}\| < \delta$ then $|E(h)| < \varepsilon$.



[Same works for $E: \mathbb{R}^n \to \mathbb{R}^m$.]

and if $F: \mathbb{R}^2 \to \mathbb{R}$ then

$\displaystyle\lim_{\vec{x}\to\vec{a}} F(\vec{x}) = c$ if $F(\vec{a}+\vec{h}) = c + E(\vec{h})$

where $E(\vec{h}) \to 0$ as $\vec{h} \to \vec{0}$.

<u>Ex</u>:

$$\lim_{(x,y)\to(0,0)} x + 3y + x^2 = 0$$

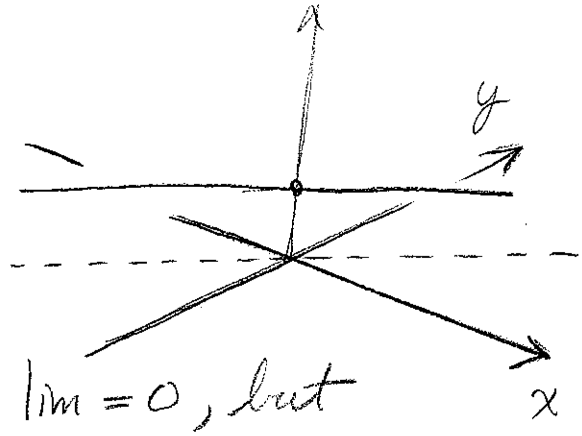
A more subtle example:

What is

$$\lim_{(x,y)\to\vec{0}} \frac{2xy}{x^2+y^2} = f(x,y) \; ?$$

not defined at $\vec{0}$ but who cares?

Approach: look at "slices"



Along x-axis:

$$f(x,0) = \frac{2x\cdot 0}{x^2+0^2} = 0$$

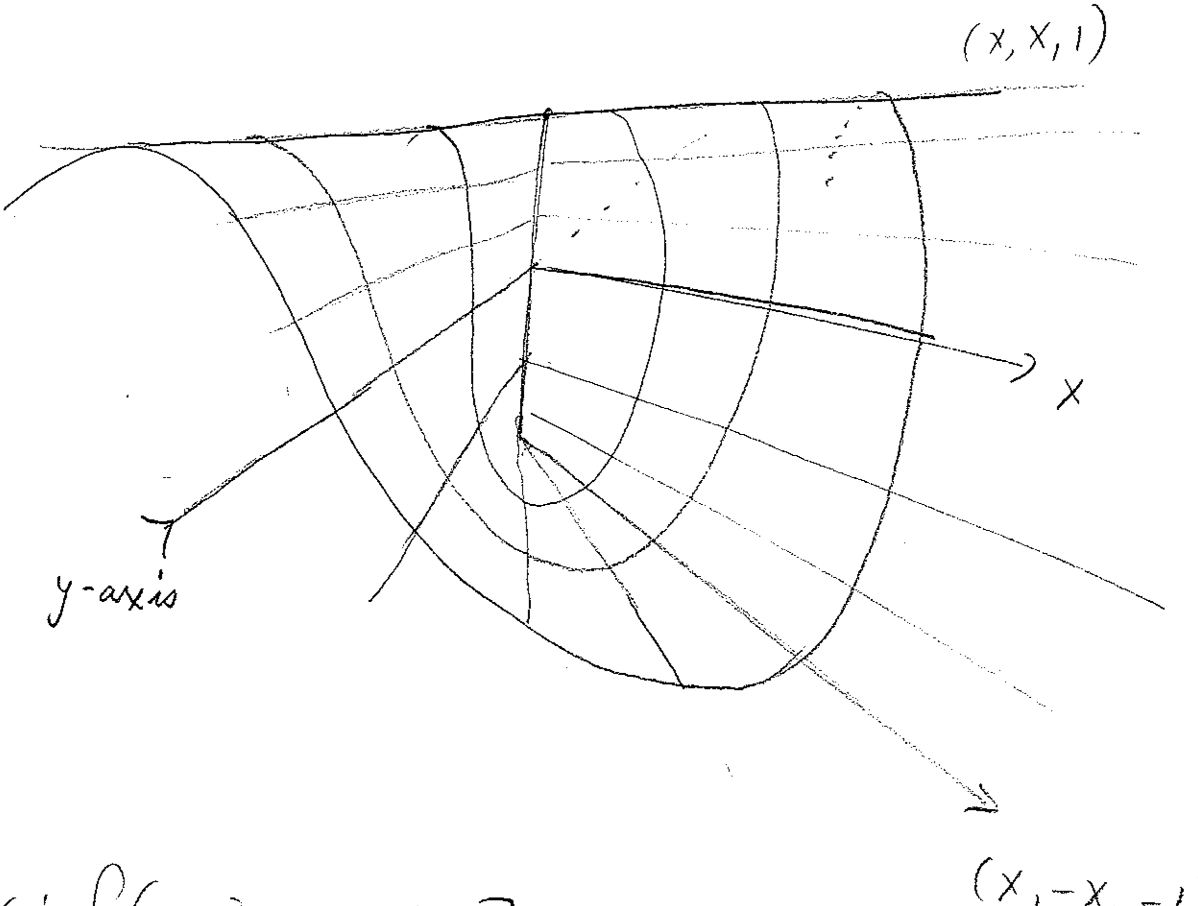
Same along y-axis. So looks like $\lim = 0$, but now look at the line $y=x$.

$$f(x,x) = \frac{2x\cdot x}{x^2+x^2} = 1 \;(?!)$$

So this limit does not exist! What about $x=-y$?

$$f(x,-x) = \frac{-2x^2}{x^2+x^2} = -1.$$ Actually, the graph of this function isn't so bad. Consider a general line $y=cx$

$$f(x,cx) = \frac{2c}{1+c^2}$$



$\underline{Ex}$: $f(x,y) = \dfrac{xy^2}{x^2+y^4}$

What is $\lim_{(x,y)\to 0} f(x,y)$? In fact

it does not exist dispite

that the limit along any line is 0....