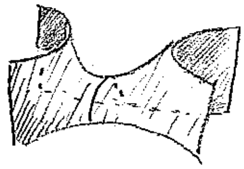
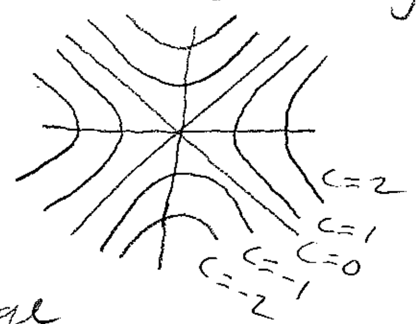
Lecture 9: Level sets in $\mathbb{R}^3$; review of limits.

<u>Last time:</u> $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = x^2 - y^2$





$f(z) = c$

<u>HW:</u> See webpage

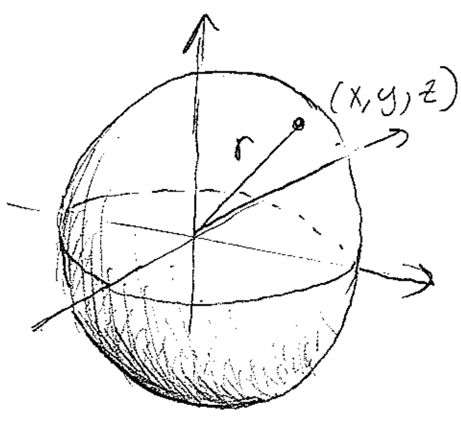
<u>Next time:</u> Rest of Section 2.3

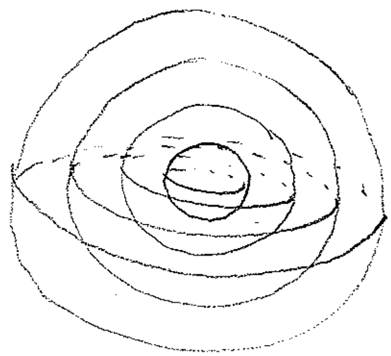
[The book has many more examples of level curves of functions of two variables.]

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ [Graph is in $\mathbb{R}^4$, but can still talk about the level sets]

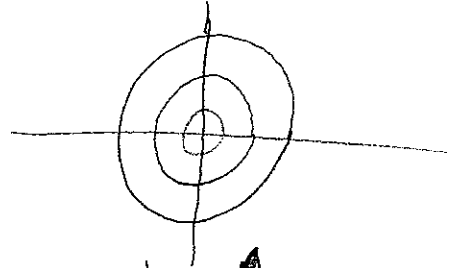
$\{ f(x,y,z) = c \}$

<u>Ex:</u> $f(x,y,z) = x^2 + y^2 + z^2$ $\qquad \{f = 1\}$ = sphere about $\vec{O}$ of radius 1.

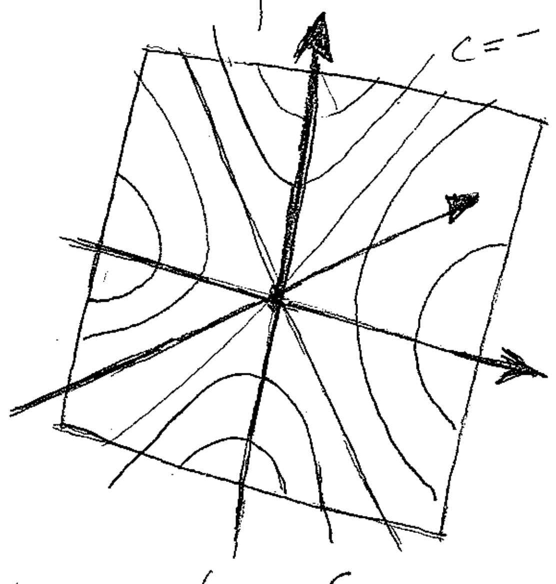




Compare $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = x^2 + y^2$



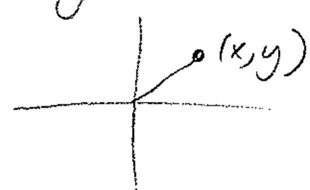
Ex: $f(x,y,z) = x^2 + y^2 - z^2$

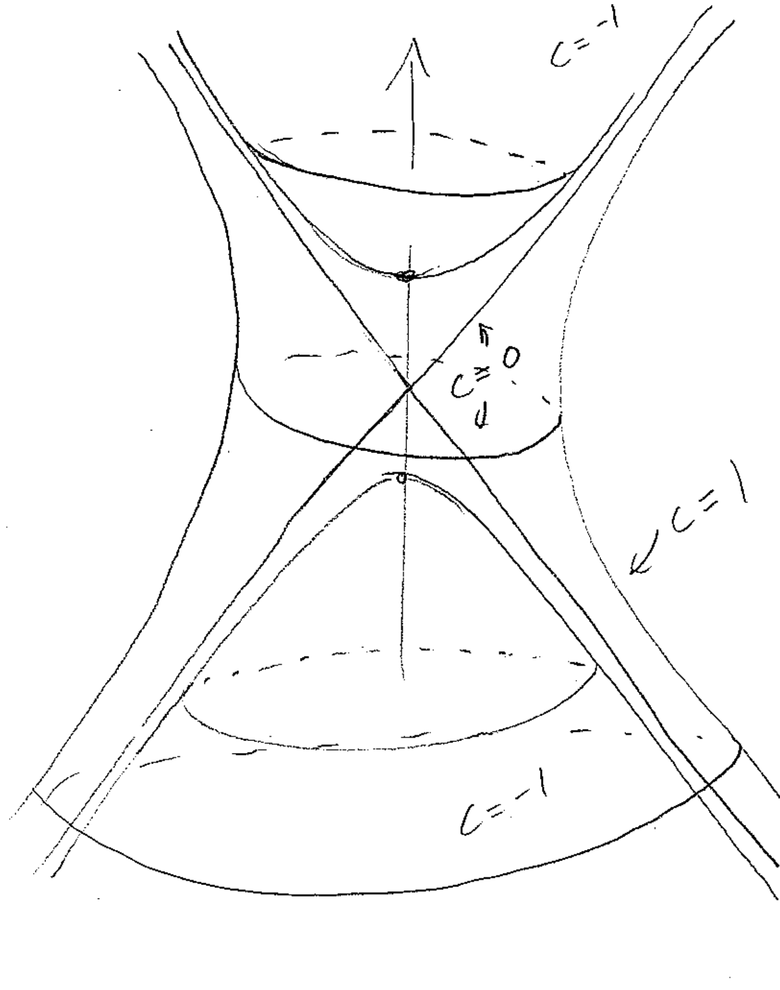


First, look at the (x,z)-plane where

$$f(x,0,z) = x^2 - z^2$$

so the level sets here are like last time. Notice there's a symmetry about the z-axis since $x^2 + y^2 = r^2$

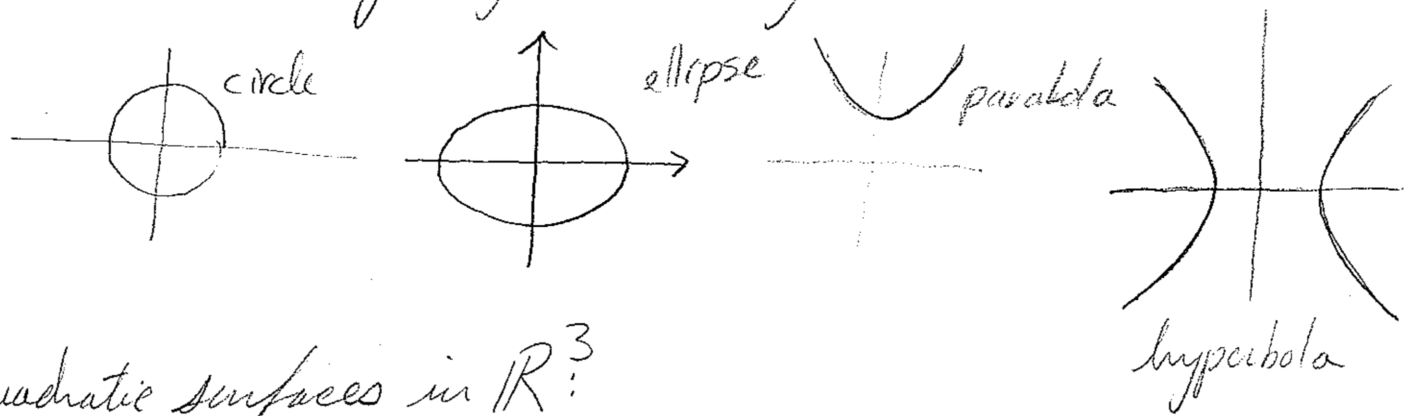




All these level sets are examples of quadric surfaces.

Conic sections: In $\mathbb{R}^2$ solutions to

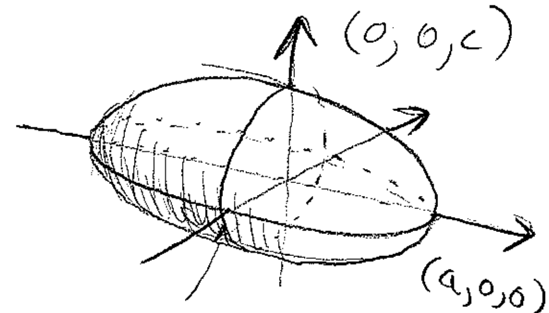
$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$



Quadratic surfaces in $\mathbb{R}^3$:

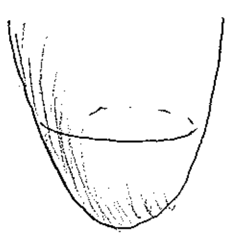
$$Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Iz+J=0$$

Ex: Ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$



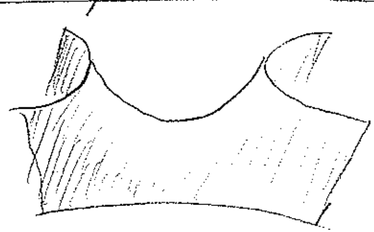
This is the image of unit sphere $x^2+y^2+z^2$ under the linear transformation $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$

Elliptic paraboloid:



$$\frac{z}{c}=\frac{x^2}{a}+\frac{y^2}{b}$$

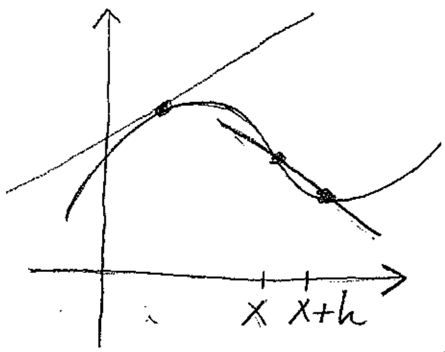
Hyperbolic paraboloid:



$$\frac{z}{c}=\frac{x^2}{a}-\frac{y^2}{b}$$

The other quadric surfaces are the double-cone and hyperboloids that we just saw as level sets.

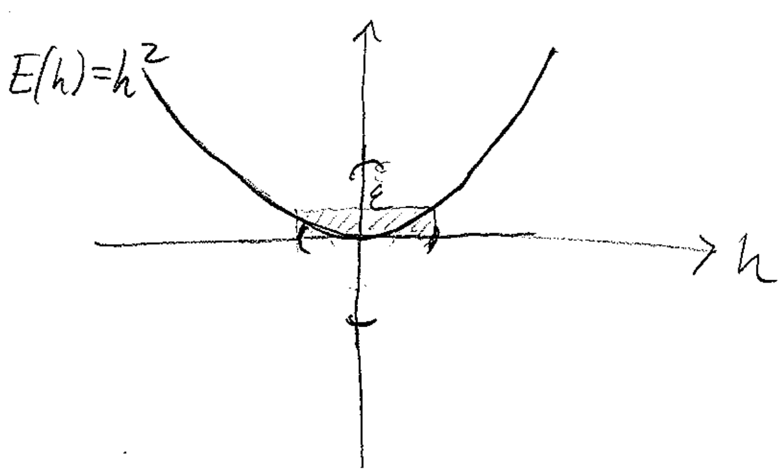
<u>Limits</u> (Section 2.3)



To talk about derivatives, we need to have limits for functions of several variables. Lets review for one variable

$$f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

<u>Def</u>: Consider $E : \mathbb{R} \to \mathbb{R}$ defined near 0. Then $\lim_{h \to 0} E(h) = 0$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that when $|h| < \delta$ then $|E(h)| < \varepsilon$



Ex: Lets show $\lim_{h \to 0} h^2 = 0$

Suppose you give me $\varepsilon > 0$. I'll take $\delta = \sqrt{\varepsilon}$

Then if $|h| < \delta$, then $|h^2| = |h|^2 < \delta^2 = \varepsilon$.

In general, we say

$$\lim_{x \to a} f(x) = C \text{ if } f(a+h) = C + E(h)$$

$$\text{where } \lim_{h \to 0} E(h) = 0.$$