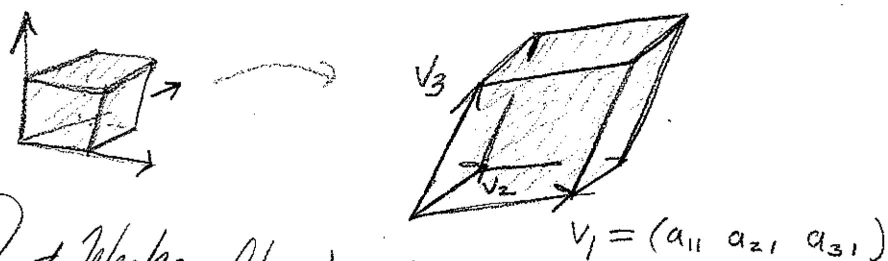
Lecture 7: Cross product (§1.5)

Last time:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

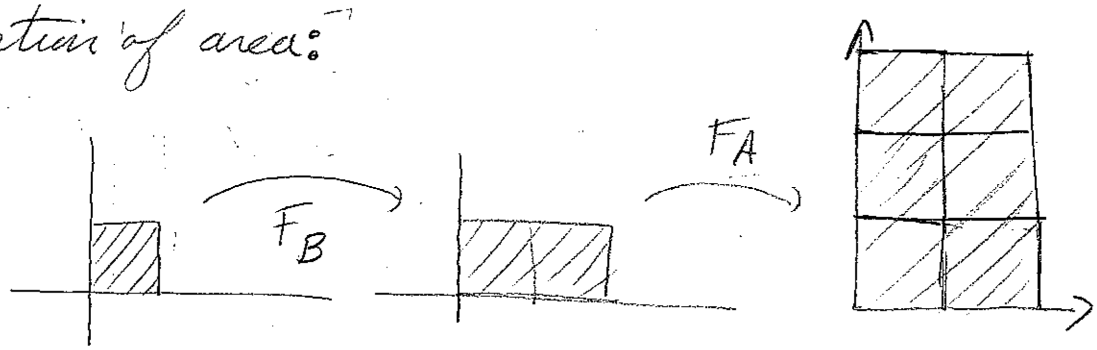
This is the volume of



Next Week: Chapter 2

HW: Handout / web. 1.5 # 6, 16, 20

One last thing about the det. If A, B are square matrices of the same size, then $\det(AB) = \det(A)\det(B)$. [You'll check this on the HW for 2×2 matrices. Can be understood in terms distortion of area:]



$B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad$ $A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad$ $AB = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$

det 2 $\quad$ det 3 $\quad$ det 6

Cross Product (c.f. dot product) For vectors in $\mathbb{R}^3$

$$\vec{v} = (v_1, v_2, v_3) = v_1\vec{i} + v_2\vec{j} + v_3\vec{k} \qquad \vec{w} = (w_1, w_2, w_3)$$

$$\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\vec{k}$$

Ex: $\vec{v} = (1, 0, 2)$ $\quad\vec{w} = (2, 1, -1)$

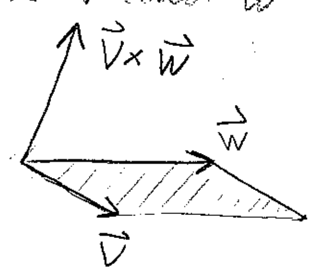
$$\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}\vec{k}$$

$$= -2\vec{i} + 5\vec{j} + \vec{k} = (-2, 5, 1)$$

Usefulness: Eg. Maxwell's Laws

Properties: 1) $\vec{v}\times\vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$

2) Its length is $\|\vec{v}\|\,\|\vec{w}\|\sin\theta$.
= area of parallelogram spanned by $\vec{v}$ and $\vec{w}$.



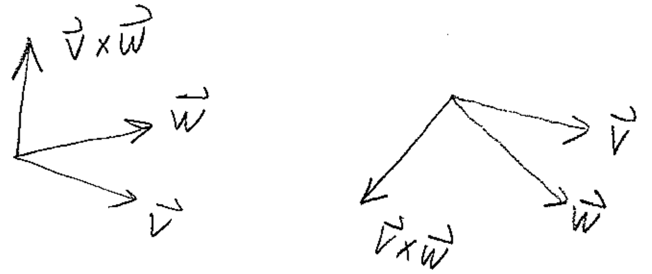
3) $\vec{v}\times\vec{w} = -\vec{w}\times\vec{v}$

4) $\vec{v}\times\vec{v} = 0$

5) $(\vec{u}+\vec{v})\times\vec{w} = \vec{u}\times\vec{w} + \vec{v}\times\vec{w}$

[Properties (1) and (2) determine $\vec{v} \times \vec{w}$ up to sign.]

Right-hand rule



Explaining the properties:

5) Calculation.

4) The first term in $\vec{v} \times \vec{v}$ is $\begin{vmatrix} v_1 & v_2 \\ v_1 & v_2 \end{vmatrix} \vec{i} = 0.$

$$\begin{vmatrix} v_1 & v_2 \\ v_1 & v_2 \end{vmatrix} = v_1 v_2 - v_1 v_2 = 0$$

3) This corresponds to switching rows

$$\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \longleftrightarrow \begin{vmatrix} w_1 & w_2 \\ v_1 & v_2 \end{vmatrix}$$

$$v_1 w_2 - v_2 w_1 \qquad v_2 w_1 - v_1 w_2$$

which changes the sign of the determinant.

1) Here's they're at right angles:

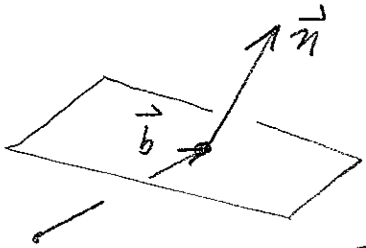
$$(\vec{v} \times \vec{w}) \cdot \vec{v} = \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} v_1 + \cdots = \begin{vmatrix} v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0$$

because of the repeat row.

2) See text.

[Very similar, but a little messier, than our calculation that $\det(A)$ = area of parallelogram.]

__Application:__ Finding equations for planes



$\rightsquigarrow$ equation for the plane.