<u>Lecture 6</u>: Today: Properties of matrix mult; determinants (finishing Section 1.4).

<u>Next time</u>: Cross products (Sect 1.5).

<u>HW</u>: On webpage: http://durfield.info/241

— ○ —

<u>Last time</u>: Matrix multiplication !

$$\overset{A}{\text{a rows}\left\{\underbrace{\begin{pmatrix} \quad & \quad & \quad \end{pmatrix}}_{\text{b cols}}\right.}\ \left.\overset{B}{\underbrace{\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_c \\ | & | & & | \end{pmatrix}}_{\text{c cols}}}\right\}\text{b rows} = \left.\underbrace{\begin{pmatrix} | & | & & | \\ A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_c \\ | & | & & | \end{pmatrix}}_{\text{c cols}}\right\}\text{a rows}$$

<u>Ex</u>: $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$

<u>Properties</u>: 1) $A(BC) = (AB)C$ (Associative)

2) $A(B+C) = AB + AC$
$(A+B)C = AC + BC$ (Distributive)

However, typically we don't have $AB = BA$!

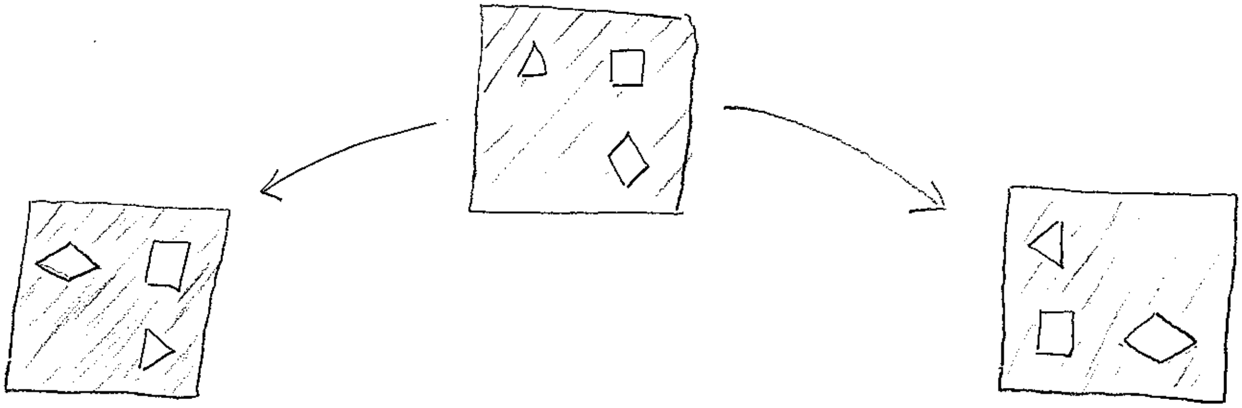
<u>Ex</u>: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\text{These differ!}\left\{\begin{aligned} AB &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ BA &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{aligned}\right.$$

In terms of linear transformations

$F_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ rotation by 90° counterclockwise

$F_B : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ reflection in the x-axis



$F_{AB} = F_A \circ F_B =$ reflection followed by rotation

$F_{BA} = F_B \circ F_A =$ rotation followed by reflection

How do these differ? Rotation by 180°. Whats the matrix for that? $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} = -I_2$ so

$$(\text{rot}_{90^\circ} \circ \text{ref}) = (\text{rot}_{180^\circ} \circ \text{ref} \circ \text{rot}_{90^\circ}) \longleftrightarrow AB = -BA$$

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<u>Determinants</u>: (square matrix) $\longrightarrow$ (number)

$$\det\begin{pmatrix}a & b\\ c & d\end{pmatrix} = ad - bc \qquad \underline{\text{Ex}}: \det\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} = 4 - 1 = 3$$

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

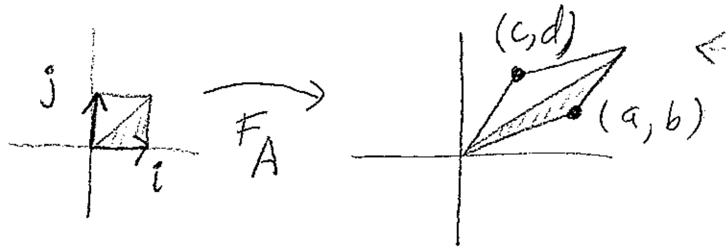
$$- a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

Ex:

means $\det\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

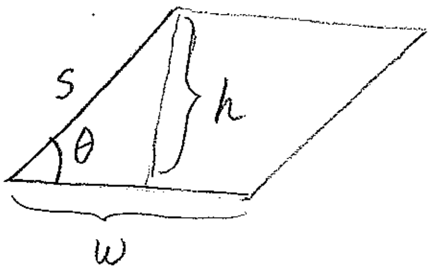
$$\det\begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{pmatrix} = 1 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} = -1 - 3 = -4$$

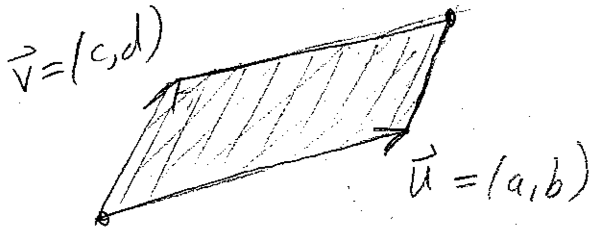
Geometric meaning: $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$



det A = Area of

Area of a pallelogram $= wh = ws \sin\theta$





$$\text{Area} = \|\vec{u}\|\,\|\vec{v}\| \sin\theta$$

This formula for area looks like our formula for $\vec{u} \cdot \vec{v}$ except we have sin not cos. [How can we go from one to the other?]

$$\text{Area}^2 = \|\vec{u}\|^2\|\vec{v}\|^2\sin^2\theta = \|\vec{u}\|^2\|\vec{v}\|^2(1-\cos^2\theta)$$

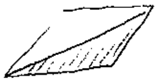
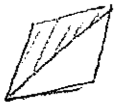
$$= \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u}\cdot\vec{v})^2 = (a^2+b^2)(c^2+d^2)-(ac+bd)^2$$

$$= \underline{a^2c^2} + a^2d^2 + b^2c^2 + \underline{b^2d^2} - \underline{a^2c^2} - \underline{b^2d^2} - 2abcd$$

$$= (ad-bc)^2 = (\det A)^2.$$

Hey, but det A can be negative!, e.g. $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

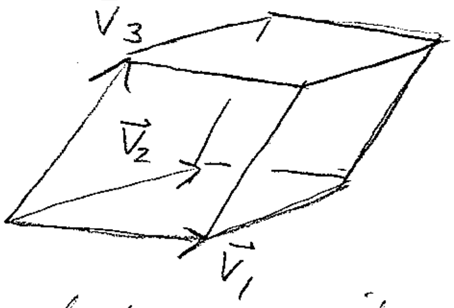
So really $|\det A| =$ Area of 

and the determinant is negative when F_A flips the square over.  vs. 

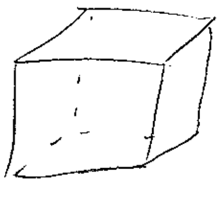
For similar reasons, if $A = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}$ is 3×3

then

$\det A =$ Volume of



which is the image of the unit cube



under F_A. If time remains, explain

$$\det AB = (\det A)(\det B)$$