Lecture 5: Today: Linear Transformations and Matrix Multiplication

HW: (Due Wed) N2: Let $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Then $F_A : \mathbb{R}^3 \to \mathbb{R}^3$ is rotation about a certain line L.
Find L and the angle through which F_A rotates.

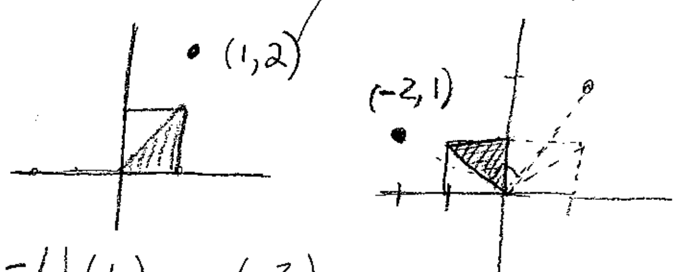
Next time: Finish Section 1.4.

Thursday: Cross product (Section 1.5)

Last time on Math 241...

A 2×2 matrix A gives a linear transformation $F_A : \mathbb{R}^2 \to \mathbb{R}^2$

Ex: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$



$$F_A((1,2)) = A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Here this is rotation by $90°$ [Counter-clockwise]

In general, an $a \times b$ matrix gives a linear transformation

$$a\{\underbrace{(\quad\quad)}_{b} \qquad F_A : \mathbb{R}^a \to \mathbb{R}^b \text{ by } F_A(\vec{v}) = A\vec{v}$$

Ex: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $F_A : \mathbb{R}^3 \to \mathbb{R}^3$ gives reflection in the xy-plane

Ex: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $F_A : \mathbb{R}^3 \to \mathbb{R}^2$ is projection

Ex: $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$ $F_A : \mathbb{R}^2 \to \mathbb{R}^3$

These all have the prop that

$$F_A(\alpha\vec{v}+\beta\vec{w}) = \alpha F_A(\vec{v}) + \beta F_A(\vec{w}) \qquad \alpha,\beta \text{ are in } \mathbb{R}$$
$$\vec{v},\vec{w} \text{ are vectors in } \mathbb{R}^b$$

Conversely, anything sat this condition comes from some matrix.

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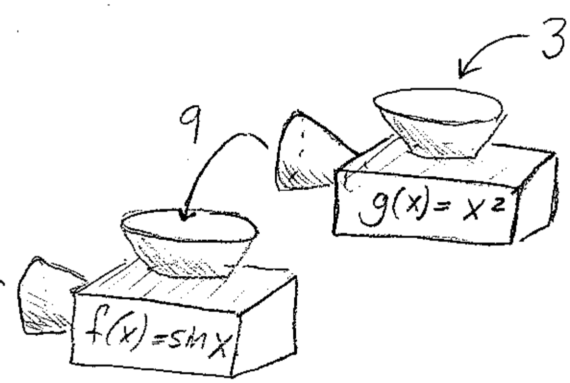
[Next, I'll explain how we can (sometimes) multiply matrices. This mysterious looking rule.]

<u>Composition of functions</u>:

$h(x) = \sin(x^2)$
$\quad = f(g(x))$

or $h = f \circ g$

$\sin(9) \approx 0.412\ldots$



[Where did you use this in single variable calc? The chain rule!]

Now consider $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$

$$F_A : \mathbb{R}^2 \to \mathbb{R}^2 \qquad F_B : \mathbb{R}^3 \to \mathbb{R}^2$$

[Note that $F_B \circ F_A$ doesn't make sense]

Let $F_A \circ F_B(\vec{v}) = F_A(F_B(\vec{v}))$ a function $\mathbb{R}^3 \to \mathbb{R}^2$.

$$F_A \circ F_B\left(\begin{pmatrix}1\\0\\1\end{pmatrix}\right) = F_A\left(F_B\left(\begin{pmatrix}1\\0\\1\end{pmatrix}\right)\right) = F_A\left(\begin{pmatrix}2\\1\end{pmatrix}\right) = \begin{pmatrix}1&2\\2&1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}4\\5\end{pmatrix}$$

Now $F_A \circ F_B(\vec{v}+\vec{w}) = F_A\left(F_B(\vec{v}+\vec{w})\right) = F_A\left(F_B(\vec{v}) + F_B(\vec{w})\right)$

$$= F_A(F_B(\vec{v})) + F_A(F_B(\vec{w})) = (F_A \circ F_B)(\vec{v}) + (F_A \circ F_B)(\vec{w})$$

[So if I was right earlier, $F_A \circ F_B$ must come from a matrix]

In fact

$$F_A \circ F_B = F_C \text{ where } C = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 2 & 3 \end{pmatrix} = AB$$

<u>Quick check</u>: $C\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 2 & 3 \end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}4\\5\end{pmatrix}$ ✓

<u>How matrix mult works:</u>

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} - \vec{u}_1 - \\ - \vec{u}_2 - \end{pmatrix}\begin{pmatrix} | & | & | \\ w_1 & w_2 & w_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} \vec{u}_1 \cdot \vec{w}_1 & \vec{u}_1 \cdot \vec{w}_2 & \vec{u}_1 \cdot \vec{w}_3 \\ \vec{u}_2 \cdot \vec{w}_1 & \vec{u}_2 \cdot \vec{w}_2 & \vec{u}_2 \cdot \vec{w}_3 \end{pmatrix}$$

In general,

$$a\left\{ \underbrace{\overset{A}{\begin{pmatrix} — \vec{u}_1 — \\ — \vec{u}_2 — \\ \vdots \\ — \vec{u}_a — \end{pmatrix}}}_{b} \circ \underbrace{\overset{B}{\begin{pmatrix} | & & | \\ \vec{w}_1 & \cdots & \vec{w}_c \\ | & & | \end{pmatrix}}}_{c} \right\} b = \overbrace{\begin{pmatrix} \vec{u}_i \cdot \vec{w}_j \end{pmatrix}}^{c} \Big\} a$$

How does this connect to our $F_A \circ F_B$ picture?

Suppose $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ $\quad \vec{V} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$

$$F_A \circ F_B(\vec{v}) = A(B\vec{v}) = A\left(\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \right)$$

$$= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} V_1 + b_{12} V_2 \\ b_{21} V_1 + b_{22} V_2 \end{pmatrix} = \begin{pmatrix} a_{11}(b_{11}V_1 + b_{12}V_2) + a_{12}(b_{21}V_1 + b_{22}V_2) \\ \text{some other mess} \end{pmatrix}$$

$$= \begin{pmatrix} (a_{11}b_{11} + a_{12}b_{21})V_1 + (a_{11}b_{12} + a_{12}b_{22})V_2 \\ \text{mess} \end{pmatrix}$$

So if the matrix of $F_A \circ F_B$ is $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$

we have $F_A \circ F_B(\vec{v}) = C\vec{V} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} c_{11}V_1 + c_{12}V_2 \\ c_{21}V_1 + c_{22}V_2 \end{pmatrix}$

so $c_{11} = a_{11}b_{11} + a_{12}b_{21} = (1^{st} \text{ row of } A) \cdot (1^{st} \text{ column of } B)$

$\quad c_{12} = a_{11}b_{12} + a_{12}b_{22} = (1^{st} \text{ row of } A) \cdot (2^{nd} \text{ column of } B)$

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