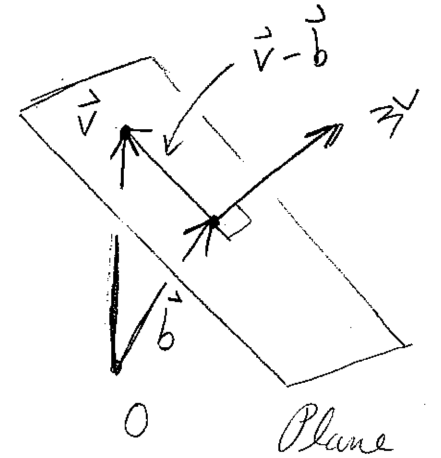
Lecture 4: Today: Matrices and Linear Transformations (§1.4)

HW: Section 1.4: 14, 15

N1: Describe the action of the linear transformations F_A and F_B where $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

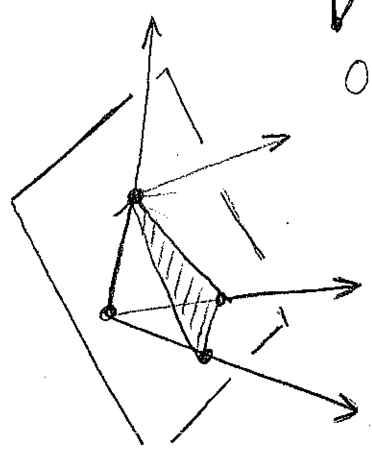
Planes in $\mathbb{R}^3$ [Application of dot product]



Can specify a plane by giving a point on it and a normal vector

$\vec{v}$ is in this plane exactly when

$$\vec{n} \cdot (\vec{v} - \vec{b}) = 0$$



Ex: $\vec{n} = (1,1,1)$ $\quad \vec{b} = (0,0,1)$

$\vec{v} = (x, y, z)$

$$0 = (1,1,1) \cdot (x, y, z-1) = x + y + z - 1 \iff x + y + z = 1.$$

Matrices:

An $m \times n$ matrix is an array of numbers with m - rows and n - columns

$$\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & \boxed{3} \\ 1 & \boxed{7} \\ 0 & \boxed{3} \end{pmatrix} \qquad \begin{pmatrix} \boxed{1 \;\; 2 \;\; 3} \\ 5 \;\; 7 \;\; 1 \end{pmatrix}$$

row

column

$2 \times 2 \qquad 3 \times 2 \qquad 2 \times 3$

Uses: Solving linear eqns
Linear transformations
optimizing functions of several variables

Basic operations:

1) Can add matrices of the same shape

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

[Sat the usual prop of addition, e.g. is commutative.]

2) Scalar mult $3\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix}$

Notation: [Tend to use capital letters for matrices.]

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = (a_{ij})$$

which row
which column

Ex: $A = (a_{ij})$, $B = (b_{ij})$ and set $C = A + B$. Then $C = (c_{ij})$ where

$$c_{ij} = a_{ij} + b_{ij}$$

Some, like the text, use square brackets: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Vectors as matrices:

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ column vector

$\vec{v} = (1, 2, 3)$ in $\mathbb{R}^3$

$(1 \ \ 2 \ \ 3)$ row vector

[In fact, can multiply matrices, start with a simple case...]

Multiplying a matrix and a column vector

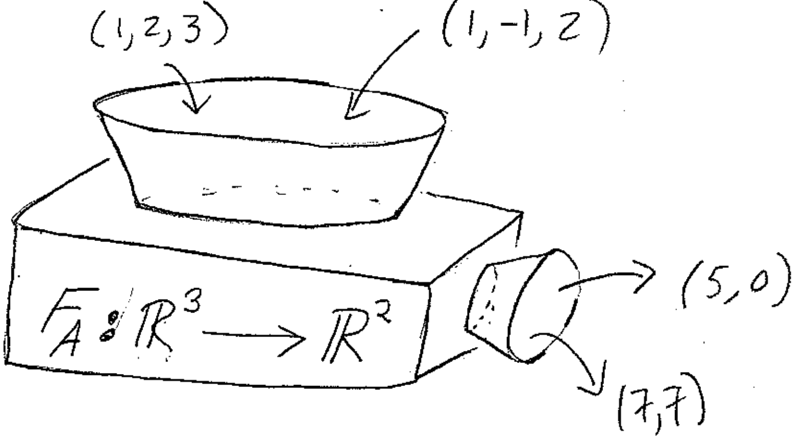
$$\underset{m \times n \text{ matrix}}{\begin{pmatrix} \text{---} & \vec{w}_1 & \text{---} \\ \text{---} & \vec{w}_2 & \text{---} \\ & \vdots & \\ \text{---} & \vec{w}_m & \text{---} \end{pmatrix}} \underset{n \times 1}{\begin{pmatrix} | \\ \vec{v} \\ | \end{pmatrix}} = \underset{m \times 1}{\begin{pmatrix} \vec{w}_1 \cdot \vec{v} \\ \vec{w}_2 \cdot \vec{v} \\ \vdots \\ \vec{w}_m \cdot \vec{v} \end{pmatrix}}$$

Ex:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{array}{c} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix} \end{array} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1+0+6 \\ 0+4+3 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}}_{A} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

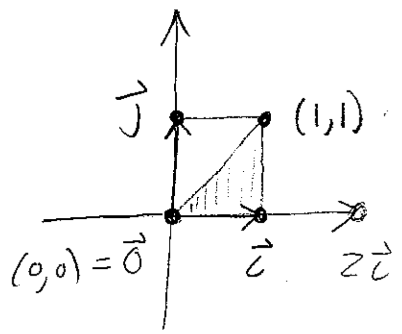
So this gives a function of several variables

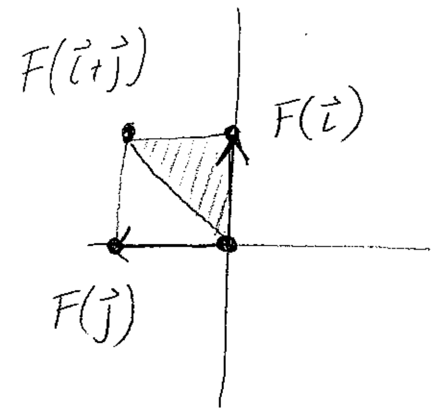


$$\boxed{F_A(\vec{v}) = A\vec{v}}$$

F_A is a linear transformation, the simplist kind of such a function.

Ex: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad F_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$





$F_A(\vec{i}) = A\vec{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{j}$ $\qquad F_A(\vec{0}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$F_A(\vec{j}) = A\vec{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\vec{i}$ $\qquad F_A((1,1)) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Properties of matrix/vector mult

1) $A(\alpha \vec{v}) = \alpha (A\vec{v})$ so know where the axis go.

2) $A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w}$

Important: So once we know $F_A(\vec{i})$ and $F_A(\vec{j})$ we know everything.

So F_A is rotation by $90°$ anticlockwise.

If time remains, do further examples, e.g. $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$