Lecture 3: Today: Applications of the dot product

HW: Sec 1.3: 14, 15, 21, 26

Next time: Matrices and Linear Transformations (Sec 1.3)

Last time: Dot product

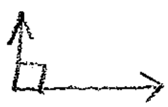
In $\mathbb{R}^3$:

$\vec{w} = (w_1, w_2, w_3)$

$\vec{v} = (v_1, v_2, v_3)$

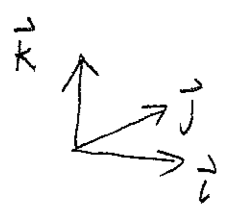
$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$
$$= \|\vec{v}\| \, \|\vec{w}\| \cos\theta$$

Vectors $\vec{v}$ and $\vec{w}$ are orthogonal if they meet at right angles, equiv $\vec{v} \cdot \vec{w} = 0$.

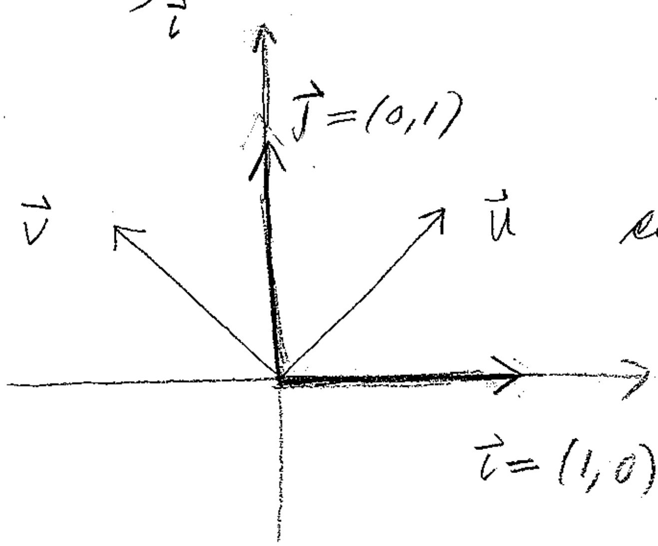


Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are orthonormal if $\vec{v}_i \cdot \vec{v}_j = 0$ and $\|\vec{v}_i\| = 1$.

Ex: In $\mathbb{R}^3$: $\vec{i} = (1,0,0)$ $\vec{j} = (0,1,0)$ $\vec{k} = (0,0,1)$

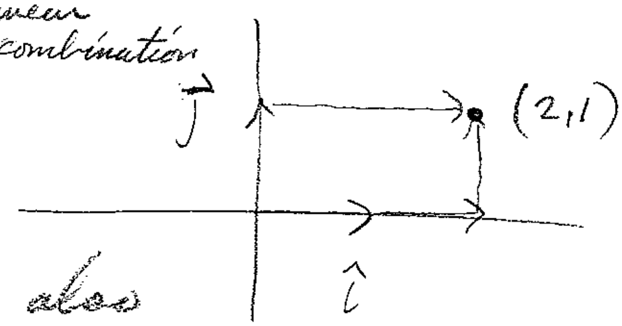


Ex: In $\mathbb{R}^2$ take $\vec{u} = (1/\sqrt{2}, 1/\sqrt{2})$
$v = (-1/\sqrt{2}, 1/\sqrt{2})$

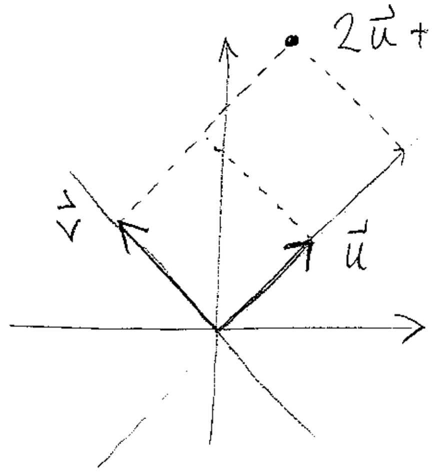


We can think of the usual coordinates on $\mathbb{R}^2$ as coming from $\vec{i}$ and $\vec{j}$ in the sense that

$(a,b) = a\vec{\imath} + b\vec{\jmath}$ ← linear combination



Other choices of a pair of independent vectors like this also gives coordinates



• $2\vec{u}+\vec{v}$ is $(2,1)$ in $(\vec{u},\vec{v})$ coor

$(1/\sqrt{2}, 3/\sqrt{2})$ is usual $(\vec{\imath},\vec{\jmath})$ coordinates

How do we find $(\vec{u},\vec{v})$ coor from the usual ones? Eg for $(2,1)$

Method 1: $a\vec{u} + b\vec{v} = (2,1)$

$$a\,(1/\sqrt{2}, 1/\sqrt{2}) + b(-1/\sqrt{2}, 1/\sqrt{2}) = \left(1/\sqrt{2}(a-b),\ 1/\sqrt{2}(a+b)\right)$$

so need to solve $\begin{cases} \frac{1}{\sqrt{2}}(a-b) = 2 \\ 1/\sqrt{2}\,(a+b) = 1 \end{cases}$

Method 2: Use dot product: [Go back to usual coor first]

$$\vec{w} = (2,1) = a\vec{u} + b\vec{v}$$

$$\vec{w}\cdot\vec{u} = (a\vec{u} + b\vec{v})\cdot\vec{u} = a(\vec{u}\cdot\vec{u}) + b(\vec{v}\cdot\vec{u}) = a$$

$$(2,1)\cdot(1/\sqrt{2}, 1/\sqrt{2}) = \frac{2}{\sqrt{2}} + 1/\sqrt{2} = 3/\sqrt{2}$$

$$\vec{w}\cdot\vec{v} = b = (2,1)\cdot(-1/\sqrt{2}, 1/\sqrt{2}) = -1/\sqrt{2}$$

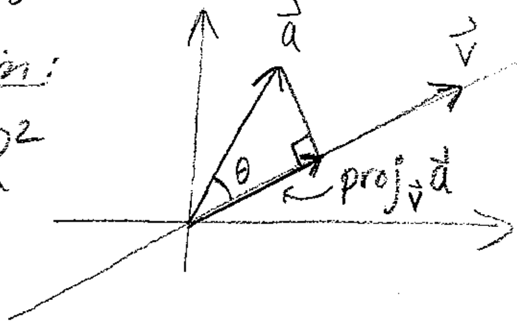
using orthonormality strongly.

Check: $(2,1) = 3/\sqrt{2}\,\vec{u} - 1/\sqrt{2}\,\vec{v}$

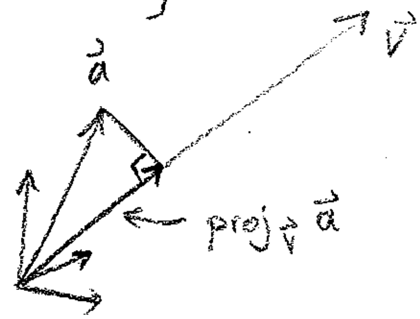
[Breaking a vector into components like this is an example of]

Projection:

In $\mathbb{R}^2$



In $\mathbb{R}^3$



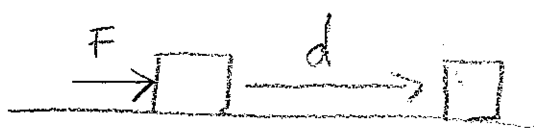
$$\|\text{proj}_{\vec{v}}\vec{a}\| = \|\vec{a}\|\cos\theta = \frac{\vec{a}\cdot\vec{v}}{\|\vec{v}\|}$$

So

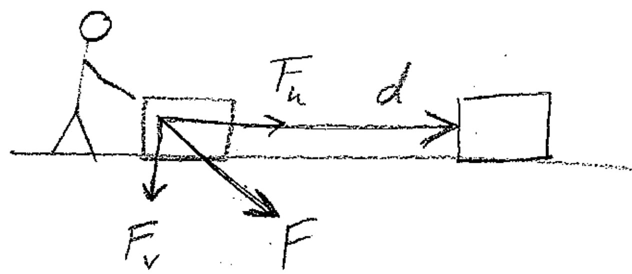
$$\text{proj}_{\vec{v}}\vec{a} = \frac{\vec{a}\cdot\vec{v}}{\|\vec{v}\|^2}\,\vec{v}$$

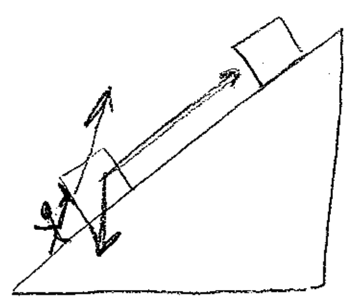
Applications:

1) Work: (force) x (distance)



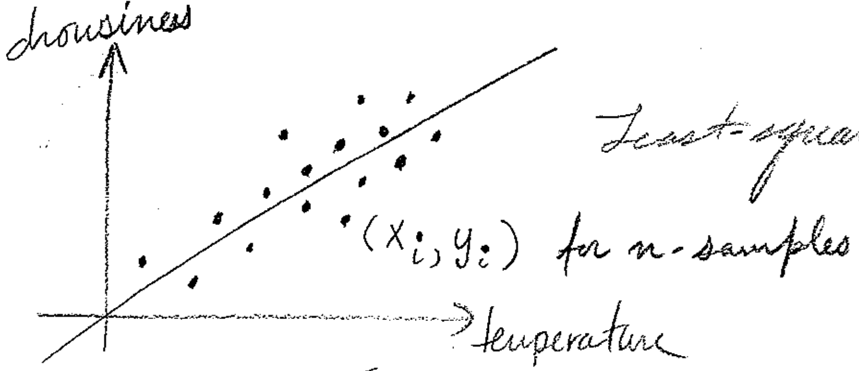
$$W = \|F_n\|\ \|\vec{d}\|$$
$$= \|\text{proj}_{\vec{d}}\vec{F}\|\ \|\vec{d}\|$$
$$= \frac{\vec{F}\cdot\vec{d}}{\|\vec{d}\|}\|\vec{d}\| = \vec{F}\cdot\vec{d}$$





2) Regression:

[Sketch]



Least-squares fit

Roughly,

$y = cx$

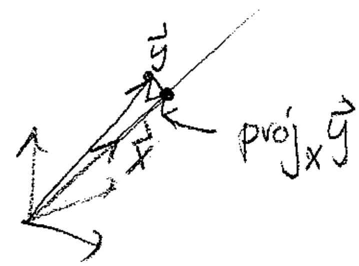
In $\mathbb{R}^n$ consider

$\vec{x} = (x_1, x_2, x_3, \dots)$

$\vec{y} = (y_1, y_2, y_3, \dots)$

If $y_i = c x_i$ then $\vec{y} = c\,\vec{x}$

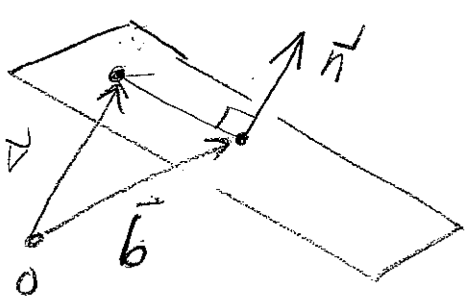
$\mathbb{R}^n$



So "best fit" is $c = \dfrac{\vec{x} \cdot \vec{y}}{\| \vec{x} \|}$

which minimizes $\| \vec{y} - c\,\vec{x} \|$. [In general, the model has more parameters, and projection is onto a plane or similar — more on this if you take linear algebra.]

3) Planes from $\perp$ vectors $\vec{n} \cdot (\vec{v} - \vec{b})$



See Example 1.26 on page 27.