Lecture 15: Derivative Miscellanea

Reminder: Exam in class on Thursday, covering through Section 2.7. Extra office hours: Friday 2-5.

HW: Sect 2.7
1, 2, 6, 28, 40

Last time: $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ given by
$$f(x,y) = (\cos y + x^2, e^{x+y})$$
$$g(u,v) = (e^{u^2}, u - \sin v)$$

General Chain Rule: $g : \mathbb{R}^n \to \mathbb{R}^m$, $f : \mathbb{R}^m \to \mathbb{R}^k$

Suppose g is diff at $\vec{x}_0$ and f is diff at $g(\vec{x}_0)$.
Then $h = f \circ g$ is diff at $\vec{x}_0$ with

$$[Dh(\vec{x}_0)] = [Df(g(\vec{x}_0))][Dg(\vec{x}_0)]$$

[Rehash moral.]

Ex $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ from last time.

$$[D(f \circ g)(\vec{0})] = [Df(\underbrace{g(\vec{0})}_{(1,0)})][Dg(\vec{0})]$$

$$= \begin{pmatrix} 2 & 0 \\ e & e \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ e & -e \end{pmatrix}$$

Feel free to check directly!

Another way of thinking about

$g: \mathbb{R} \to \mathbb{R}^2 \qquad f: \mathbb{R}^2 \to \mathbb{R} \qquad h = f \circ g : \mathbb{R} \to \mathbb{R}$

$g(t) = (x(t), y(t)) \qquad\qquad h(t) = f(x(t), y(t))$

$$\frac{dh}{dt} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} =$$

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$ ← x and y are playing two roles.

[In words: The rate h is changing w.r.t to t is the rate f changes w.r.t to x times the rate x is changing...]

Sometimes folks elide the distinction between f and h and write $\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt}$.

——— ∘ ———

Alternate notations

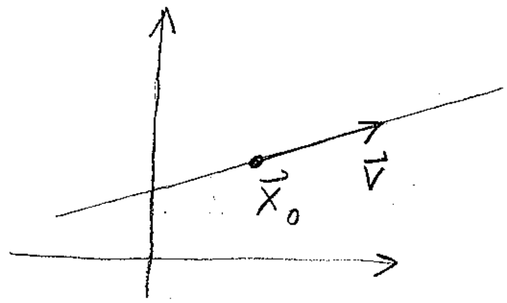
$$\frac{\partial f}{\partial x} = f_x = D_1 f \qquad \frac{\partial f}{\partial y} = f_y = D_2 f$$

[Turn now to Section 2.7, which has some important topics for min/max.]

Directional Derivatives: $f: \mathbb{R}^2 \to \mathbb{R}$

[Already have ∂-derivatives, measuring change in (x,y) directions. But why give the axes special treatment.]

Pick $\vec{x}_0 \in \mathbb{R}^2$, $\vec{v}$ a vector. Then the derivative of f in direction $\vec{v}$ at $\vec{x}_0$ is



$D_{\vec{v}} f =$ rate of change in f as we move along $\vec{v}$

$$= \frac{d}{dt} \underbrace{f(\vec{x}_0 + t\vec{v})}_{\text{fn of one var}} \Big|_{t=0}$$

$\boxed{\text{Also denoted } \nabla_{\vec{v}}}$

Ex: $\vec{v} = (1,0)$ then $D_{\vec{v}} f(\vec{x}_0)$ is just $\frac{\partial f}{\partial x}(\vec{x}_0)$

In general, can find via the chain rule

$$g: \mathbb{R} \to \mathbb{R}^2 \quad g(t) = (f \circ g)'(0) = D(f \circ g)(0)$$

$$= Df(g(\vec{0}) = \vec{x}_0)\, Dg(0)$$

$$= \left(\frac{\partial f}{\partial x}(\vec{x}_0) \quad \frac{\partial f}{\partial y}(\vec{x}_0) \right) \vec{v} = \frac{\partial f}{\partial x}(\vec{x}_0) v_1 + \frac{\partial f}{\partial y}(\vec{x}_0) v_2$$

since if $\vec{x}_0 = (x_0, y_0)$, $\vec{v} = (v_1, v_2)$

$$g(t) = (x_0 + t v_1, y_0 + t v_2)$$

$$g'(t) = Dg = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{v}.$$

<u>Gradient:</u>

$$\underset{\substack{\parallel \\ \text{grad } f}}{\nabla f(\vec{x}_0)} = \left(\frac{\partial f}{\partial x}(\vec{x}_0), \frac{\partial f}{\partial y}(\vec{x}_0) \right)$$

[Just Df, thought of as a vector.]

So

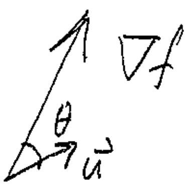
$$D_{\vec{v}} f(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \vec{v}$$

<u>Note:</u> $D_{2\vec{v}} f \overset{?}{=} \nabla f(\vec{x}_0) \cdot 2\vec{v} = 2(\nabla f(\vec{x}_0) \cdot \vec{v}) = 2 D_{\vec{v}} f$

In what direction is f increasing most?

$\vec{u}$ a unit vector, i.e. $\|\vec{u}\| = 1$

$$D_{\vec{u}} f = \nabla f(\vec{x}_0) \cdot \vec{u} = \|\nabla f(\vec{x}_0)\| \, \|\vec{u}\| \cos\theta$$



Thus the direction of greatest increase is

$$\frac{\nabla f(\vec{x}_0)}{\|\nabla f(\vec{x}_0)\|}$$ and the rate of increase is $\|\nabla f(\vec{x}_0)\|$