Lecture 13: More on derivatives (§2.4 and 2.5)

Last time: Suppose $f: \mathbb{R}^n \to \mathbb{R}^m$ is defined near $\vec{x}_0$.
The function f is differentiable near $\vec{x}_0$ if there is linear transformation $L: \mathbb{R}^n \to \mathbb{R}^m$ so that

$$f(\vec{x}_0 + \vec{h}) = f(\vec{x}_0) + L(\vec{h}) + E(\vec{h}) \text{ where } \lim_{\vec{h} \to \vec{0}} \frac{E(\vec{h})}{\|\vec{h}\|} = 0.$$

The matrix of L is denoted $Df(\vec{x}_0)$.

HW: §2.4 44, 62, 64. §2.5 2, 3, 10, 39

Next Time: §2.6

Component Functions:

Ex: $f: \mathbb{R} \to \mathbb{R}^2$ $\quad f(2) = (1, 3)$ ← entries are functions of input

$$f(x) = (f_1(x), f_2(x)) \text{ where } f_1, f_2 : \mathbb{R} \to \mathbb{R}$$

In general, $f: \mathbb{R}^n \to \mathbb{R}^m$ have $f(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x}))$

where $f_i : \mathbb{R}^n \to \mathbb{R}$

Formula for $Df(\vec{x}_0)$ [aka "The Jacobian"]

$$Df(\vec{x}_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\vec{x}_0) & \frac{\partial f_1}{\partial x_2}(\vec{x}_0) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{x}_0) \\ \frac{\partial f_2}{\partial x_1}(\vec{x}_0) & \cdots & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\vec{x}_0) & \cdots & & \frac{\partial f_m}{\partial x_n}(\vec{x}_0) \end{pmatrix}$$

<u>How to remember:</u>

$$Df(\vec{x}_0)\vec{h} \approx \begin{array}{c}\text{change in } f \\ \text{as we move} \\ \text{from } \vec{x}_0 \text{ to } \vec{x}_0+\vec{h}\end{array} = \begin{pmatrix} \Delta f_1 \\ \Delta f_2 \\ \vdots \\ \Delta f_m \end{pmatrix}$$

So row i of Df should only involve f_i. Take $\vec{h} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ to see column i should only involve $\frac{\partial}{\partial x_i}$

<u>Thm</u>: Differentiable functions are continuous

<u>Continuous</u>: $f(\vec{x}_0 + \vec{h}) = f(\vec{x}_0) + \underbrace{E(\vec{h})}$ where $\lim_{\vec{h}\to 0} E(\vec{h}) = 0$

<u>Diff</u>: $f(\vec{x}_0 + \vec{h}) = f(\vec{x}_0) + \overbrace{Df(\vec{x}_0)(\vec{h}) + \bar{E}(\vec{h})}$ where $\lim_{\vec{h}\to 0} \frac{\bar{E}(\vec{h})}{\|\vec{h}\|} = 0$

As linear transformations are continuous, the 2^{nd} notion is a refinement of the first.

When is a function differentiable?

Need at least that $\frac{\partial f_i}{\partial x_j}$ all exist at $\vec{x}_0$

Not enough as consider $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

This vanishes along both axes so $\frac{\partial f}{\partial x}(\vec{0}) = \frac{\partial f}{\partial y}(\vec{0}) = 0.$

But f isn't continuous at $\vec{0}$!

<u>Useful Crit</u>: If $\frac{\partial f_i}{\partial x_j}$ exist and are continuous near $\vec{x}_0$ then f is differentiable at $\vec{x}_0$.

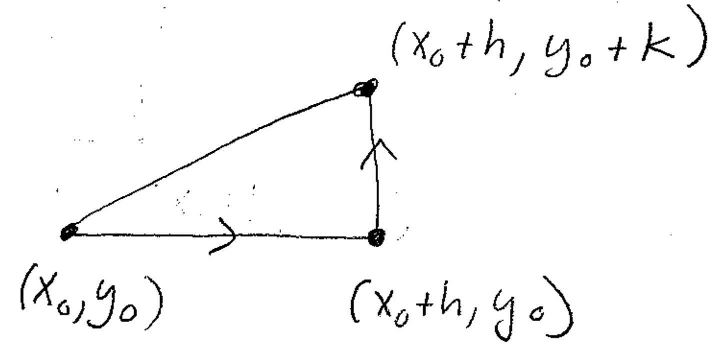
<u>Ex</u>: $f(x,y) = x^3 y \sin(xy^2)$ is diff on all of $\mathbb{R}^2$.

Reason crit works is a little complicated (see text) but is related to:

$f(x_0+h, y_0+k) \approx$

$f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)h$

$+ \frac{\partial f}{\partial y}(x_0+h, y_0)k$



wrong input, but $\frac{\partial f}{\partial y}$ is cont

so this doesn't matter much.

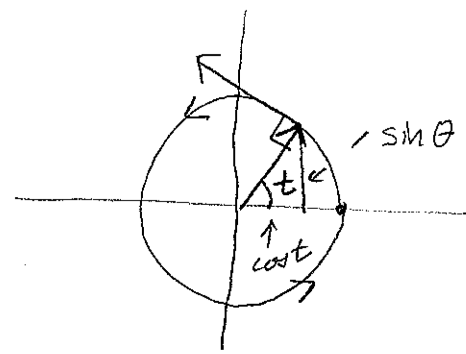
Ex: Parametric curves

$$\vec{c}: \mathbb{R} \longrightarrow \mathbb{R}^2 \text{ or } \mathbb{R}^3$$

For instance

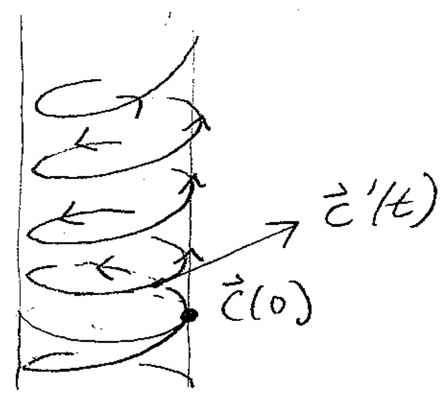
$$\vec{c}(t) = (c_1(t), c_2(t)) = (\cos t, \sin t)$$

$$D\vec{c}(t) = \begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$



which is basically the velocity vector to the curve. Sometimes denote as $c'(t)$ [and write as a row vector.]

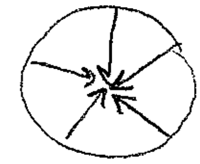
$$\vec{c}(t) = (\cos t, \sin t, t)$$



$$D\vec{c}(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}$$

[Draw from top to bottom.]

For such a path, $\vec{c}'(t)$ is velocity and $\vec{c}''(t)$ is the acceleration. Eg in the first example $\vec{c}''(t) = \begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix} = -\vec{c}(t)$,



which is why you have to exert centripetal force to keep objects moving in a circle $\boxed{\vec{F}(t) = m\,\vec{c}''(t)}$

Ex: $f: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad f(x,y) = (\underbrace{\cos y + x^2}_{f_1}, \underbrace{e^{x+y}}_{f_2})$

$$Df(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & -\sin y \\ e^{x+y} & e^{x+y} \end{pmatrix}$$

$$Df((1,0)) = \begin{pmatrix} 2 & 0 \\ e & e \end{pmatrix}$$

$g: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad g(u,v) = (e^{u^2}, u - \sin v) = (g_1, g_2)$

$$Dg(u,v) = \begin{pmatrix} 2ue^{u^2} & 0 \\ 1 & +\cos v \end{pmatrix} \quad Dg(0,0)$$

Consider $f \circ g: \mathbb{R}^2 \to \mathbb{R}^2$

$$f \circ g(u,v) = f(g(u,v)) = f(e^{u^2}, u - \sin v)$$

$$= \left(\cos(u - \sin v) + e^{2u^2}, e^{e^{u^2} + u - \sin v}\right)$$

compute $D(f \circ g)(\vec{0})$ ...

there has got to be a better way.

Where have you seen this before???