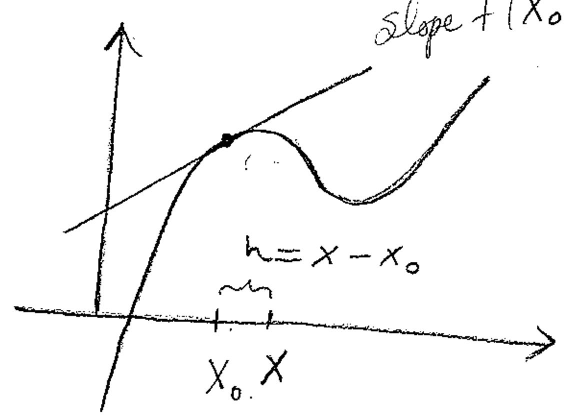
Lecture 12: Derivatives (§2.4)

HW: (Due Feb 12) Section 2.4 # 13, 23, 29, 38

Next time: Rest of §2.4; §2.5.

Reminder: First exam is Thursday, Feb 14.

Derivatives:



One var: $f : \mathbb{R} \to \mathbb{R}$

The tangent line is given by

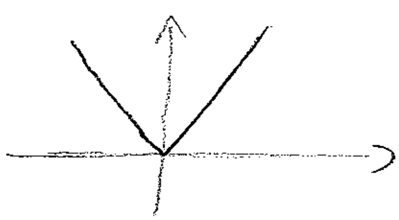
$$g(x_0+h) = f(x_0) + f'(x_0)h$$

and this "well approximates" f in the sense that

$$(*) \quad f(x_0+h) = f(x_0) + f'(x_0)h + E(h)$$

where $E(h)$ is small when h is small.

In particular, $\lim_{h\to 0} E(h) = 0$. But that's not enough, consider $f(x) = |x|$, at $x_0 = 0$. Then $f(h) = \underset{0}{\underset{\parallel}{f(x_0)}} + 0 \cdot h + E(h)$



where $E(h) = |h|$ and so $\to 0$ as $h \to 0$.

[Query: what happens if "$f'(0)$" $= 1$.] Correct condition

$$(*) \quad \lim_{h \to 0} \frac{E(h)}{h} = 0.$$

Similarly,

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + E(h)$$

where $\lim_{h\to 0} \frac{E(h)}{h^2} = 0$.

Notice that (*) leads to $\dfrac{f(x_0+h)-f(x_0)}{h} = f'(x_0) + \dfrac{E(h)}{h}$
and so taking $h \to 0$ makes the connection to what you know clear.

Suppose $f: \mathbb{R}^2 \to \mathbb{R}$. The derivative of f at $\vec{x}_0$ is a linear transformation $L: \mathbb{R}^2 \to \mathbb{R}$ s.t.

$$f(\vec{x}_0 + \underbrace{\vec{h}}_{\text{small}}) = f(\vec{x}_0) + L(\vec{h}) + E(\vec{h}) \text{ where}$$

$$\lim_{\vec{h}\to 0} \frac{E(\vec{h})}{\|\vec{h}\|} = 0.$$

"Best Linear Approximation"

[Same works for any function $f: \mathbb{R}^n \to \mathbb{R}^m$.]

L is given by a matrix of size [Query] $(a \;\; b)$

If $\vec{h} = (h_1, h_2)$ then

$$f(\vec{x}_0 + \vec{h}) = f(\vec{x}_0) + (a \;\; b)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + E(\vec{h})$$

$$= f(x_0) + ah_1 + bh_2 + E(\vec{h})$$

To solve for a, let's take $\vec{h} = (\vec{h}_1, 0)$

$$f(\vec{x}_0 + (\vec{h}_1, 0)) = f(x_0) + ah_1 + E(\vec{h})$$

$$\shortparallel$$

$$f(x_0 + h_1, y_0) \quad \text{So } a = \frac{f(x_0+h_1, y_0) - f(x_0)}{h_1} - \frac{E(\vec{h})}{h_1}$$

where $\vec{x}_0 = (x_0, y_0)$

and

$$a = \lim_{h_1 \to 0} \frac{f(x_0+h_1, y_0) - f(x_0, y_0)}{h_1}$$

which is $\frac{\partial f}{\partial x}(\vec{x}_0)$ the partial derivative of f with respect to x. As y_0 is fixed this is really just a one-var derivative.

The other entry of the matrix is $b = \frac{\partial f}{\partial y}(\vec{x}_0)$.

Ex: $f(x,y) = x^3 y \sin(xy^2)$

$$\frac{\partial f}{\partial x} = 3x^2 y \sin(xy^2) + \overbrace{x^3 y \left(\cos(xy^2)\right) \cdot y^2}^{x^3y^3\cos(xy^2)}$$

$$\frac{\partial f}{\partial y} = x^3 \sin(xy^2) + \underbrace{x^3 y \cos(xy^2) \cdot (2xy)}_{2x^4y^2\cos(xy^2)}$$

So we should be able to approximate f near $\vec{x}_0 = (1,1)$ by $L: \mathbb{R}^2 \to \mathbb{R}$ given by

$$\begin{pmatrix} 3\sin(1) + \cos(1) & \sin(1) + 2\cos(1) \end{pmatrix} = \begin{pmatrix} 3.06 & 1.92 \end{pmatrix}$$

When $f: \mathbb{R}^n \to \mathbb{R}^m$ is defined on a ball near $\vec{x}_0$, we say it is differentiable there when we can find a linear trans. $L: \mathbb{R}^n \to \mathbb{R}^m$ where

$$f(\vec{x}_0+\vec{h}) = f(\vec{x}_0) + L(\vec{h}) + E(\vec{h}) \text{ where } \lim_{\vec{h}\to 0} \frac{E(\vec{h})}{\|\vec{h}\|} = 0.$$

Where L is denoted $Df(\vec{x}_0)$

[The matrix for L is given by partial derivatives as I'll describe later. For now lets continue to focus on $\mathbb{R}^2 \to \mathbb{R}$.]

When is f differentiable at $\vec{x}_0$? [Query.] Need $\frac{\partial f}{\partial x}(\vec{x}_0)$ and $\frac{\partial f}{\partial y}(\vec{x}_0)$ to exist.

But that's not enough, e.g. $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

At $\vec{0}$, both partials are 0, but f isn't even continuous at $\vec{0}$ [and thus not even diff.]

<u>Useful fact:</u> If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are <u>continuous</u> near $\vec{x}_0$ then f is differentiable there.

<u>Ex</u>: $f(x,y) = x^3 y \sin(xy^2)$ is differentiable on all of $\mathbb{R}^2$

<u>Thm</u>: Differentiable functions are continuous.

Continuous means $f(\vec{x}_0+\vec{h}) = f(\vec{x}_0) + E(\vec{h})$ where $E(\vec{h}) \to 0$ as $\vec{h} \to \vec{0}$.

Diff means $f(\vec{x}_0+\vec{h}) = f(\vec{x}_0) + \overbrace{Df(\vec{x}_0)\vec{h} + F(\vec{h})}$ where $\frac{F(\vec{h})}{\|\vec{h}\|} \to 0$ as $\vec{h} \to \vec{0}$

As linear transformations are continuous, the 2nd notion is a refinement of the other.