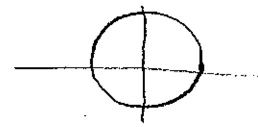
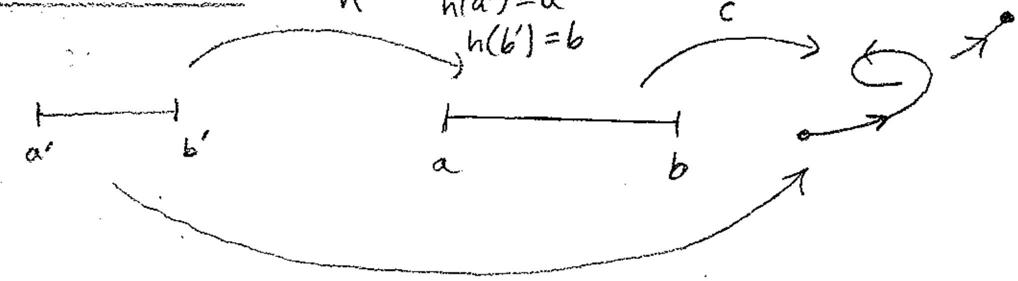
Lecture 26: More on integration over paths.

Last time: paths vs. curves

$c(t) = (\cos t, \sin t)$



Reparametrization: h — monotone increasing ($h' > 0$)
$h(a') = a$
$h(b') = b$



$d = c \circ h$

HW:

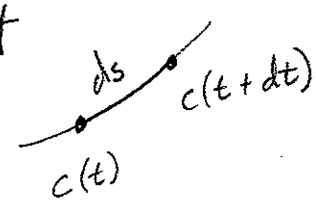
Next time: § 5.4

Office Hours for Friday: Cancelled, but plenty next week...

C a curve in the plane, param by $c : [a,b] \to \mathbb{R}^2$

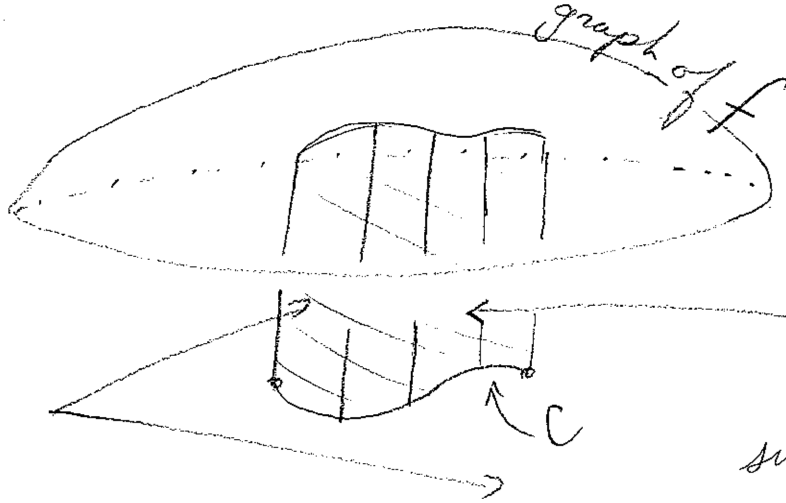
$f : \mathbb{R}^2 \to \mathbb{R}$

Define
$$\int_C f \, ds = \int_a^b f(c(t)) \underbrace{\|c'(t)\| \, dt}_{\text{length element}}$$

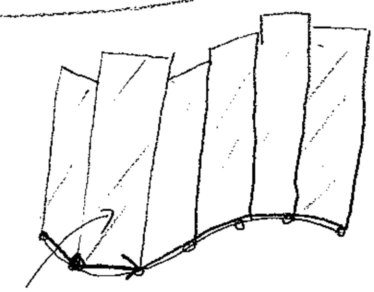


[Last time we saw an interpretation in terms of averages. Here's another point of view.]

$\int_C f\, ds =$ area of



since



Last time, I botched taking averages because I didn't account for the param.

$$\text{Area} = f(c(t_i))\,\|c(t_{i+1})-c(t_i)\|$$
$$\approx f(c(t_i))\,\|c'(t_i)\|\,\Delta t$$

Here we're measuring something "geometric" so $\int_C f\,ds$ shouldn't depend on the parameterization.

But lets check... $d = c\circ h$ another param of C

$$\int_{a'}^{b'} f(d(t))\,\|d'(t)\|\,dt = \int_{a'}^{b'} f(c(h(t)))\,\|c'(h(t))\|\cdot h'(t)\,dt$$

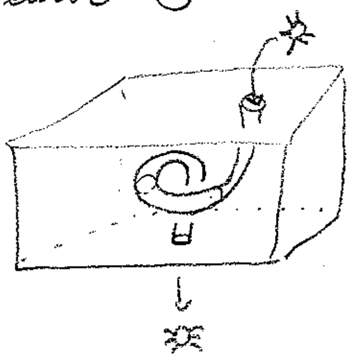
$$u = h(t)$$
$$du = h'(t)\,dt$$

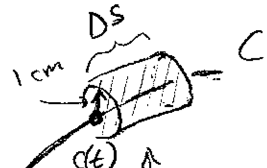
$$d'(t) = c'(h(t))\cdot h'(t) \qquad = \int_a^b f(c(u))\,\|c'(u)\|\,du.$$

So $\int_C f\,ds$ doesn't depend on the parameterization.

[Straight-forward to compute insofar as it just reduces to ordinary one var integrals.]

<u>Another use</u>: Suppose we bore a 2 cm diam tunnel in a material of density $\underbrace{\rho(x,y,z)}_{\text{units} = g/cm^3}$ along a curve C





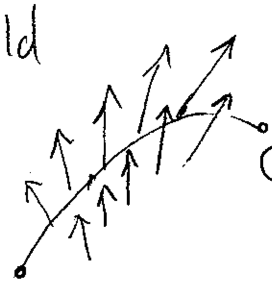
Volume is $\pi(cm)^2 \Delta s$

mass is $\rho(c(t))\, \pi\, (cm)^2 \Delta s$

Total material removed

$$= \int_C \pi\, (cm)^2 \rho(x,y,z)\, ds$$

$F : \mathbb{R}^2 \to \mathbb{R}^2$ a vector field



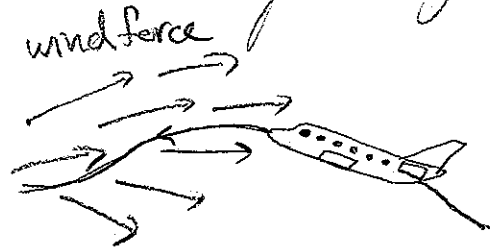
C - curve. param by $c : [a,b] \to \mathbb{R}^2$

$$\int_C F \cdot ds$$

$$= \int_a^b F(c(t)) \cdot c'(t)\, dt$$

Why consider such a quantity? $W = F \cdot d$

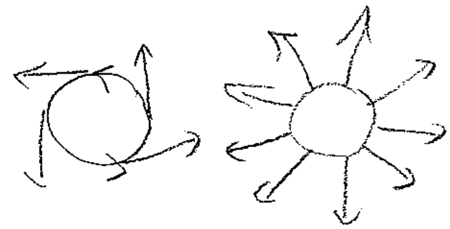
<u>Ex</u>:



Work done by wind

$$= \int_C F \cdot ds$$

Can also think of $\int_C F \cdot ds$ as the "circulation" of the vector field along C.



Why does it not depend on the parametrization?

$$\int_a^b F(c(t)) \cdot c'(t)\, dt = \int_a^b F(c(t)) \cdot \left(\frac{c'(t)}{\|c'(t)\|}\right) \|c'(t)\|\, dt$$

$$= \int_a^b F(c(t))\, u(t)\, ds$$

unit tangent vector $u(t)$

Notes: Actually two kinds of reparam.: orientation preserving and orientation reversing.

$\int_C F \cdot ds$ only invariant under the 1st kind, though $\int_C f\, ds$ is invariant under both.