Lecture 25 Integration along paths (§5.1, 5.2)

Last time: Length of a curve $= \int_a^b \|c'(t)\|\,dt$

$c:[a,b] \to \mathbb{R}^2$ 

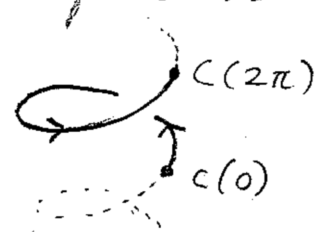
HW: On Web

Next time: Rest of §5.2, 5.3.

Same method computes the length of a curve in $\mathbb{R}^3$.

$c(t) = (\cos t, \sin t, t)$

where $0 \le t \le 2\pi$



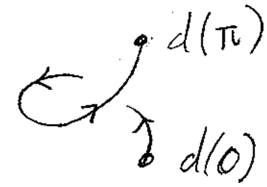
$c'(t) = (-\sin t, \cos t, 1)$

$\|c'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

So the curve has length

$$\int_0^{2\pi} \|c'(t)\|\,dt = \int_0^{2\pi} \sqrt{2}\,dt = \boxed{2\sqrt{2}\,\pi}$$

Now consider the closely related example, which traces out the same path



$d(t) = (\cos 2t, \sin 2t, 2t)$ for $0 \le t \le \pi$

Length should work out to be the same!

$d'(t) = (-2\sin 2t, 2\cos 2t, 2)$ which has length

$\sqrt{4\sin^2 2t + 4\cos^2 2t + 4} = 2\sqrt{2}$

So $$\int_0^{\pi} \|c'(t)\|\,dt = \int_0^{\pi} 2\sqrt{2}\,dt = \boxed{2\sqrt{2}\,\pi}$$

Really, we have two [closely related concepts] here.

Path: $c:[a,b]\to\mathbb{R}^2, \mathbb{R}^3$ a particular motion in space.

Curve: the points traced out by a path

A path that runs along a particular curve is called a <u>parameterization</u> of it. [Next: How diff params are related.]

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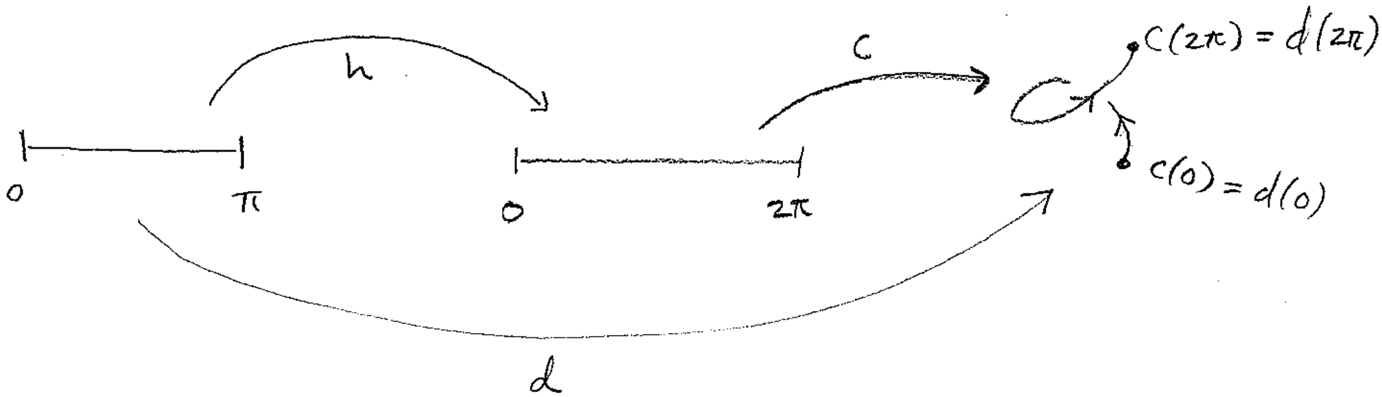


$h:[0,\pi]\to[0,2\pi]$ given by $h(t)=2t$.

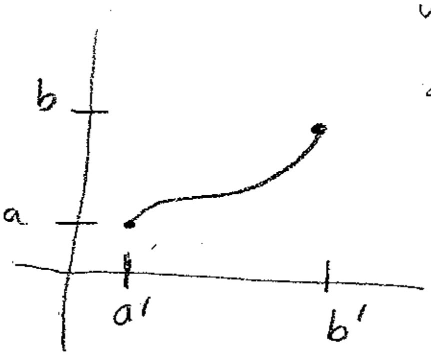
Then $d=c\circ h$ as $c\circ h(t)=c(h(t))=c(2t)=d(t)$.

<u>Reparameterization</u>: $c:[a,b]\longrightarrow\mathbb{R}^n$ a path

$h:[a',b']\longrightarrow[a,b]$ a cont function

where $h(a')=a$ $h(b')=b$

and h is increasing ($h'>0$)



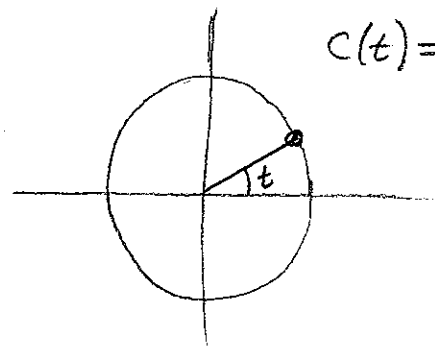
Set $d:[a',b']\to\mathbb{R}^n$

by $d=c\circ h$

This traces out the same path at [typically] differing speeds. [Do example of the circle.]

Integration over paths:

Temperature: $T(x,y) = 10x^2$

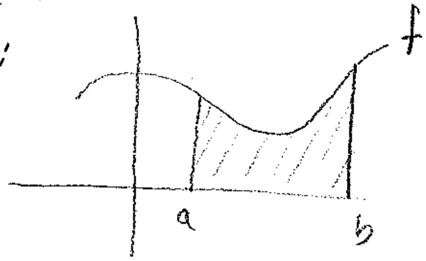


$c(t) = (\cos(t), \sin(t))$

$0 \leq t \leq 2\pi$

Q: Find the average of T on the unit circle.

Recall:



$$\text{Average of } f = \frac{1}{b-a}\int_a^b f(x)\,dx$$

This suggests:
$$\text{Average Temp} = \frac{1}{\text{length of circle}} \int_{\text{circle}} T$$

$$= \frac{1}{2\pi}\int_0^{2\pi} T(c(t))\,dt = \frac{1}{2\pi}\int_0^{2\pi} 10\cos^2(t)\,dt = 5$$

But what if we reparametrize with $d(t) = (\cos(2t), \sin(2t))$

$0 \leq t \leq \pi$

$$\text{Ave Temp} = \frac{1}{2\pi}\int_0^{\pi} T(d(t))\,dt = \frac{1}{2\pi}\int_0^{\pi} 10\cos^2 2t\,dt$$

$u = 2t$

$du = 2dt$

$$= \frac{5}{2\pi}\int_0^{2\pi} \cos^2 u\,du = \frac{5}{2}$$

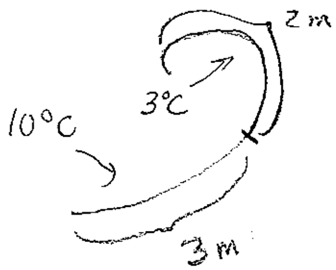
$\boxed{?!}$ How can we have gotten a different ans?

Lets do a unit analysis $\quad \sum T(c(t_i))\Delta t$

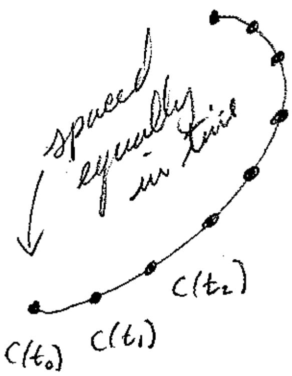
$$\approx$$

$$\underset{m \nearrow \text{length}}{\frac{1}{\text{length}}} \int \underset{°C}{\text{Temp}} \, \underset{s}{dt} = \frac{°C \, s}{m}$$

whereas this should be just °C.



Average would be $10 \cdot \frac{3}{5} + 3 \cdot \frac{2}{5} = \frac{36}{5}$ °C



Average is

$$\approx \frac{1}{\text{length}} \sum \left(\text{temp on } \overline{c(t_i)\,c(t_{i+1})}\right)\left(\text{length of } \overline{\bullet\!-\!\bullet}\right)$$

$$\approx \frac{1}{\text{length}} T(c(t_i))\|c'(t_i)\|\Delta t$$

$$\Longrightarrow \text{Average} = \frac{1}{\text{length}} \int_a^b T(c(t))\,\|c'(t)\|\,dt$$

__Check:__ T, d as above $\|d'(t)\| = 2$ so

$$\frac{1}{2\pi}\int_0^{2\pi} T(d(t))\,\|d'(t)\|\,dt = 5 \text{ as before!}$$