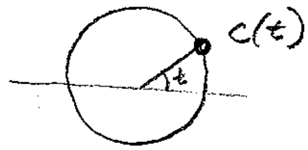
Lecture 24: Curves in Space (§3.1, 3.3)

HW:

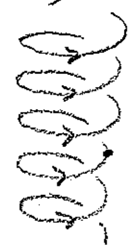
Next time: §5.1, §5.2



Curve: $c : [a,b] \longrightarrow \mathbb{R}^2$ or $\mathbb{R}^3$

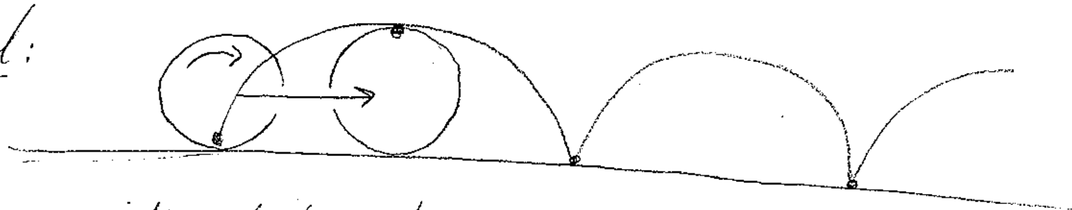
Ex: c on $[0,2\pi]$ given by $c(t) = (\cos t, \sin t)$

Ex: $c : \mathbb{R} \to \mathbb{R}^3$ given by $c(t) = (\cos t, \sin t, t)$



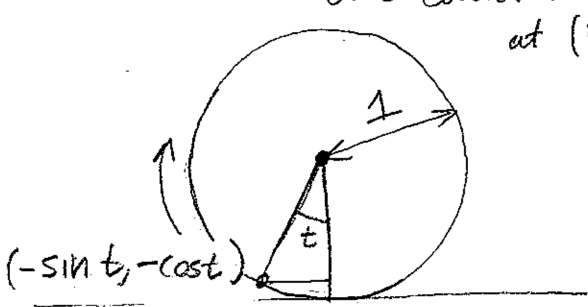
Section 3.1 gives many examples [read it!]

Cycloid:



Suppose it rotates at a rate of 1-radian per second — after time t, the center has moved a distance of t.

$t=0$ center is at $(0,1)$, at time t it is at $(t,1)$

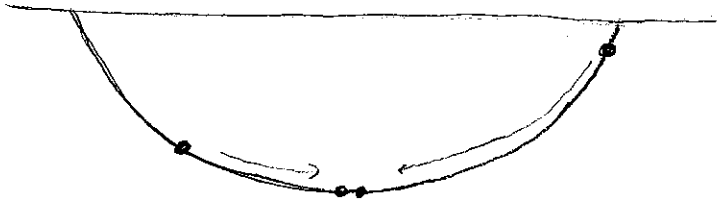


$$c(t) = (t,1) + (-\sin t, -\cos t)$$
$$= (t - \sin t, 1 - \cos t)$$

wheel rotating clockwise

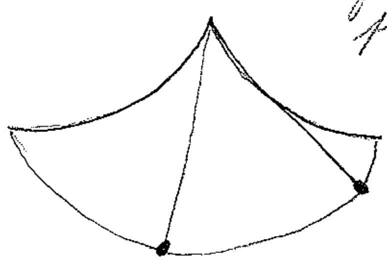
This curve has the following interesting properties

Tautochrone



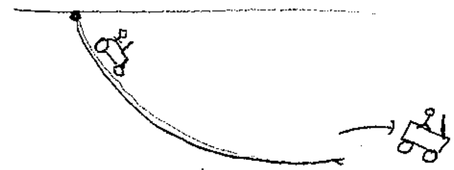
same time to the bottom

Huygens (1650s) pendulum clock.



A flexible pendulum hanging on a cycloid swings along a cycloid

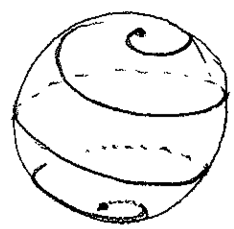
Brachistochrone



minimizes time.

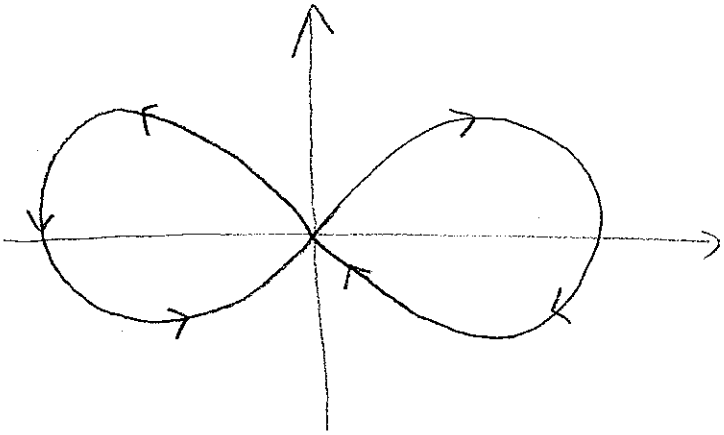
This is an example of an infinite dimension optimization problem "calculus of variations".

$\mathbb{R}^3$ examples:



Implicitly defined curves: $f:\mathbb{R}^2 \to \mathbb{R}$

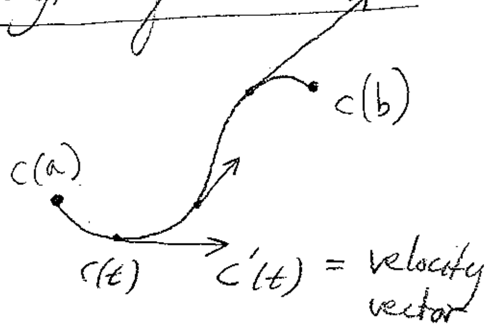
$f = 0$ Ex: $f(x,y) = (x^2+y^2)^2 - x^2 + y^2$



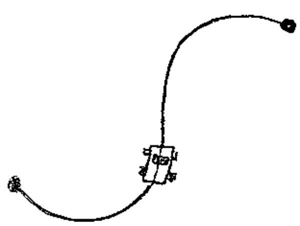
Could try to parametrize this....

lemniscate

Length of a curve: $c : [a,b] \to \mathbb{R}^2$ $[\text{or } \mathbb{R}^3]$



How long is this path?

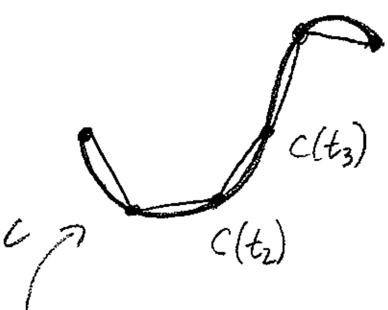


speed at time t

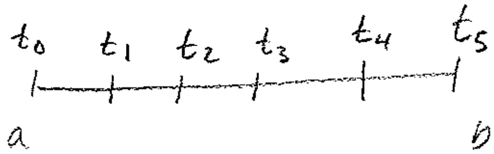
$= \|c'(t)\|$

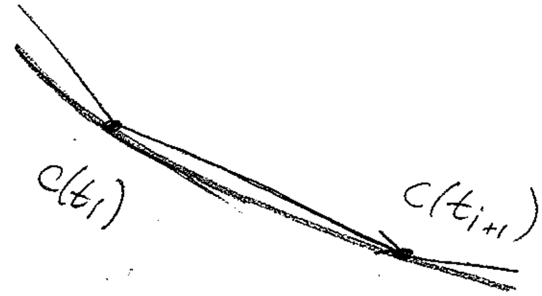
$$\text{distance traveled} = \int_a^b \text{speed}\, dt = \int_a^b c(t)\, dt$$

Another way to think about this: Can measure lengths of straight segments, so let's approximate



$$\text{Length} \approx \sum_{i=0}^{n} \|c(t_3) - c(t_2)\|$$



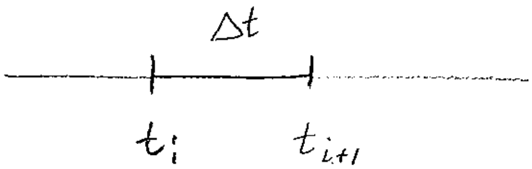


$$C(t_i + \Delta t) \approx C(t_i) + C'(t_i)\Delta t$$

$$C(t_{i+1}) - C(t_i) = C'(t_i)\Delta t$$

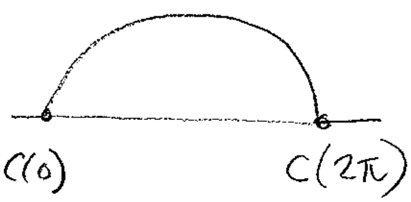
So length $\approx$

$$\sum_{i=0}^{n} \|C'(t_i)\| \, \Delta t$$



As $\Delta t \to 0$ these Riemann sums converge to $\displaystyle\int_a^b \|C'(t)\| \, dt.$

<u>Ex</u>: Cycloid



$$C(t) = (t - \sin t, 1 - \cos t)$$

$$C'(t) = (1 - \cos t, + \sin t)$$

$$\|C'(t)\| = \sqrt{2 - 2\cos t}$$

$$\text{Length} = \int_0^{2\pi} \sqrt{2}\sqrt{1 - \cos t} \, dt = \int_0^{2\pi} \sqrt{2}\sqrt{2\sin^2 \tfrac{t}{2}} \, dt$$

$$= \int_0^{2\pi} 2 \sin \tfrac{t}{2} \, dt = \int_0^{\pi} 4 \sin u \, du = 8$$

$u = t/2$

$du = \frac{1}{2} dt$