Lecture 23: Linear Programming

Next time: § 3.1, 3.3   HW: Web

---

min/max of a linear function constrained by linear (in)equalities. [Among one of the common types of min/max, w/ application to logistics, biology, etc.]

Ex: Lumber mill producing two products

	finish grade	construction grade
Profit	\$120 per thousand board ft	\$100 per T.B.F.
Time to saw	2 hours per T.B.F	same
Time to plane	5 hour per T.B.F	3 hr per T.B.F

Constraints: Saw available 8 hr/day, plane avail 15 hr/day.

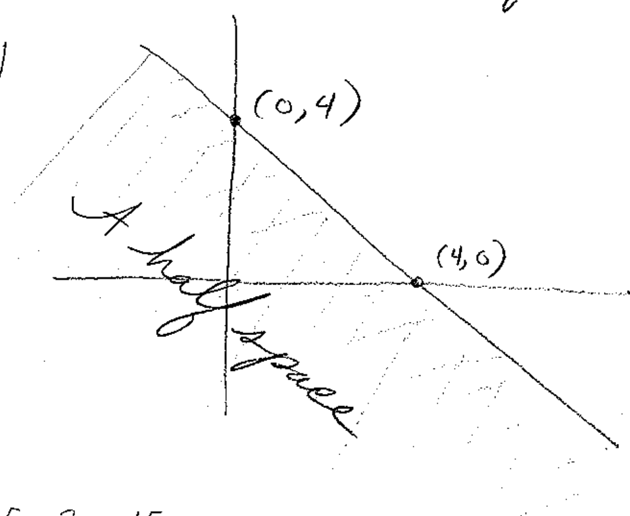
Q: What choice of products maximizes profit?

Setup: $X = \#$ of TBF of finish grade, $y = \#$ of construction grade.

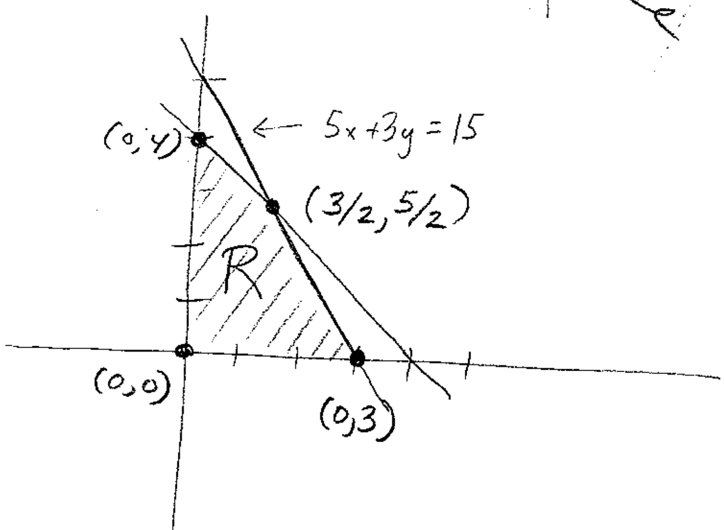
Max: $P(x,y) = 120x + 100y$

Subject to: $\begin{cases} 2x+2y \leq 8 \\ 5x+3y \leq 15 \end{cases}$ $\quad x \geq 0$, $y \geq 0$ ← negative inputs not so meaningful.

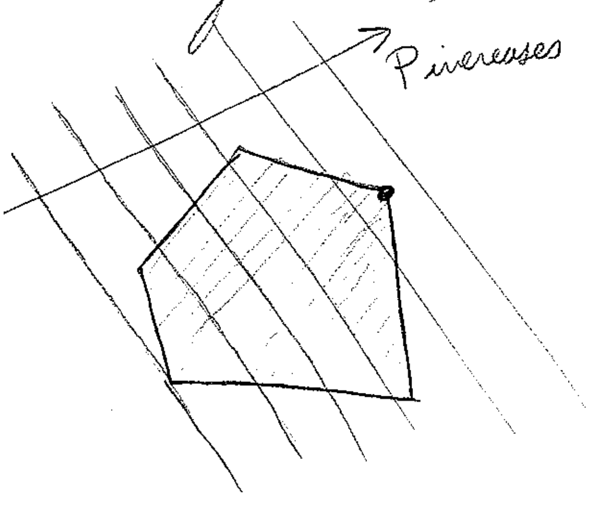
Consider 1st Constraint: $x + y \leq 4$



The region sat all the constraints is called the "feasible region"



Now P is a linear function, so its level sets look like [Query]



Key: min/max occurs at a vertex

(on interior, $\nabla P \neq 0$)

In our example $P(0,0) = 0$ $\quad P(0,4) = 400$

$P(3,0) = 360$ $\quad \boxed{P(3/2, 5/2) = 430}$ global max.

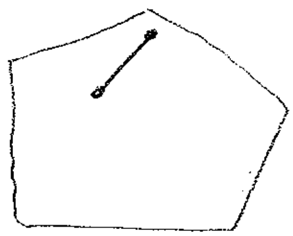
General 2-var Linear Prog: max: $c_1 x + c_2 y$

subject to: $a_{11}x + a_{12}y \geq b_1$

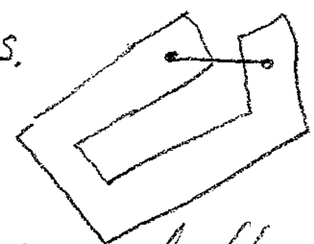
$a_{21}x + a_{22}y \geq b_2$

$\vdots$

The feasible region is always convex:



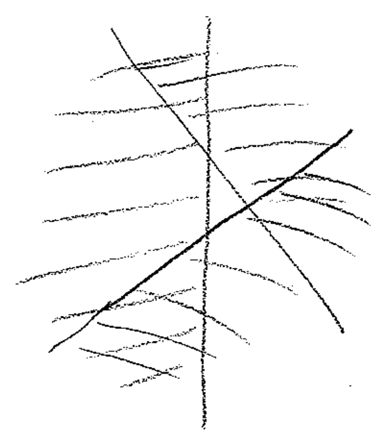
vs.



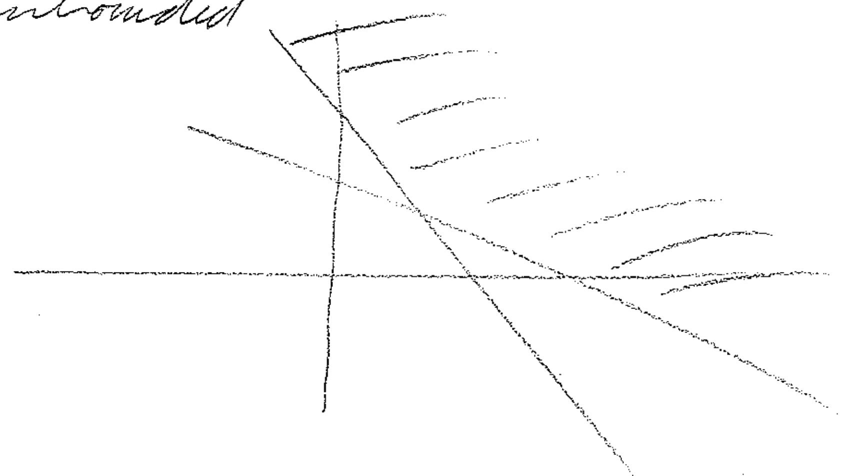
[Since it is the intersection of convex half spaces.]

and min/max occurs at some vertex.

Note: The feasible region can be empty

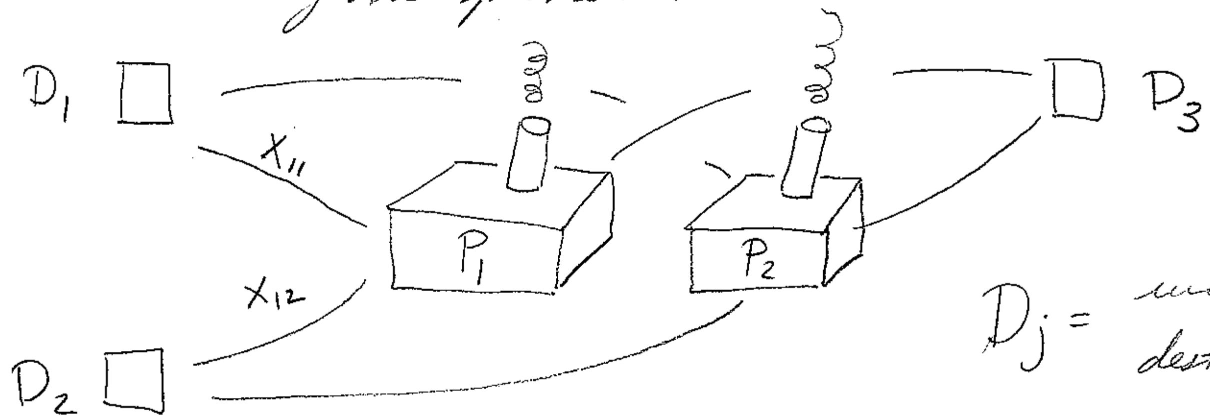


or unbounded



in which case there *may* be no max/min.

Linear programming was first studied during WWII for use in logistics problems.



D_j = widget destinations

Widget factories

X_{ij} – amount shipped from P_i to D_j (6 vars)

$$X_{11} + X_{12} + X_{13} \leq p_1 = \text{max production at } P_1$$
$$X_{21} + X_{22} + X_{23} \leq p_2 \quad \text{" " " } P_2$$
$$X_{11} + X_{21} \geq d_1 = \text{demand at } D_1$$
$$X_{12} + X_{22} \geq d_2$$
$$X_{13} + X_{23} \geq d_3$$
$$X_{ij} \geq 0$$

<u>Cost Fn:</u> $C(x_{11}, x_{12}, \ldots, x_{23}) = a_{11}X_{11} + a_{12}X_{12} + \cdots$

where a_{ij} = cost of shipping widgets from P_i to D_j

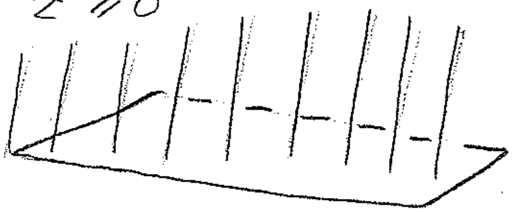
Want to minimize this.

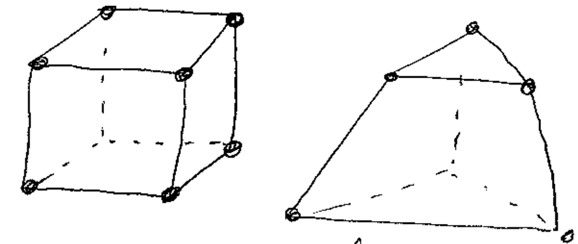
Problem: hard to visualize things in 6 dimensions.

Sol: Abstraction.

Feasible regions:
Convex polytopes

3d: $z \geq 0$

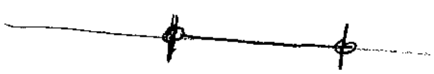
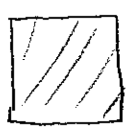
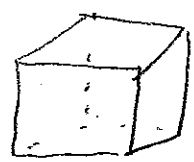




[Min/max will occur at vertices just as before.]

Problem: The number of vertices can grow rapidly with the dimension [even if the # of equations remains modest]

In $\mathbb{R}^n$: $0 \leq x_i \leq 1$

$n = 1$	2	3	4	n
			4-cube	
verts 2	4	8	16	2^n

In real life, may have 10,000+ variables

Sol: Simplex algorithm.