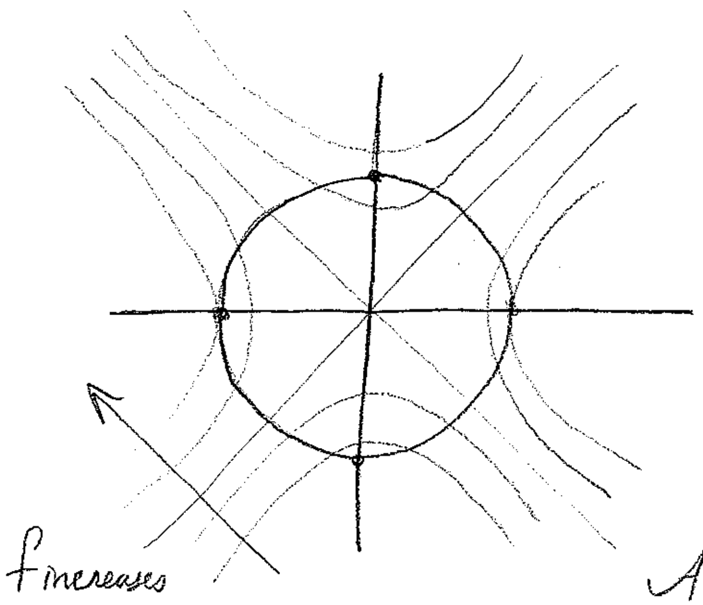
Lecture 22: Lagrange Multipliers (§4.4)

Office Hours on Friday: By appointment.

Next time: Linear Programming (not in text.)

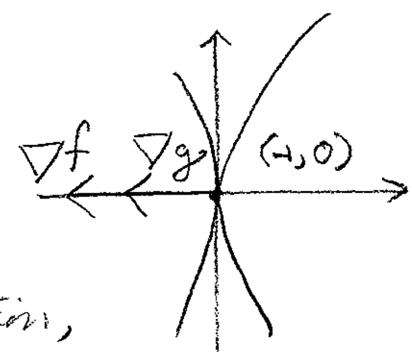
Last time: Maximize $f(x,y) = x^2 - y^2$ on the unit circle

$\{g=1\}$ where $g(x,y) = x^2+y^2$



Local min/max can only occur where the circle is tangent to the level curve of f

At such points,



∇f and ∇g point in the same direction, i.e.

$$\nabla f = \lambda \nabla g \quad \text{[Key algebraic char.]}$$

Lagrange Multipliers: [Discovered by Euler.]

Critical points are where

$$\nabla f = \lambda \nabla g \quad \text{and} \quad \left(g = 1 \iff x^2+y^2=1\right)$$

$\nabla f = (2x, -2y)$

$\nabla g = (2x, 2y)$

$$\begin{cases} x = \lambda x \\ y = -\lambda y \end{cases}$$

If $x \neq 0$, then $\lambda = 1 \Rightarrow y = 0 \Rightarrow x = \pm 1$

If $y \neq 0$, then $\lambda = -1 \Rightarrow x = 0 \Rightarrow y = \pm 1$

So the crit pts are

	$(1,0)$	$(-1,0)$	$(0,1)$	$(0,-1)$
λ	1	1	-1	-1
f	1	1	-1	-1
	global max		global min	

<u>Ex</u>: Find the point on $\{\underbrace{x - y + 2z}_{g(x,y,z)} = 3\}$ closest to $\vec{0}$

$f(x,y,z) = (\text{dist to } \vec{0})^2 = x^2 + y^2 + z^2$

<u>Critical points:</u> $\nabla f = \lambda \nabla g \qquad g = 3$

$\nabla f = (2x, 2y, 2z)$

$\nabla g = (1, -1, 2)$

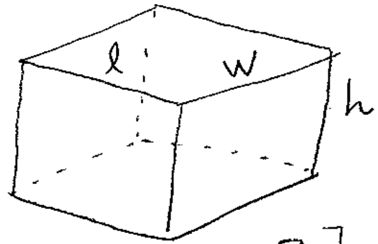
$$\Rightarrow \quad \begin{aligned} 2x &= \lambda \\ 2y &= -\lambda \\ z &= \lambda \end{aligned} \quad \Rightarrow \quad \begin{aligned} z &= 2x \\ y &= -x \end{aligned}$$

Combine with $g = 3$ gives $x + x + 4x = 3 \Rightarrow x = 1/2$

So only crit pt is $(1/2, -1/2, 1)$, just as before

[Note: 1) Algebra easier than before
2) Don't have to solve for one of the vars, which we won't be able to do for a complicated g.]

Ex: Find the rectangular box with area 6 of largest volume. [Query: what do think the answer is?]



Maximize $V(l,w,h) = lwh$ subject to

$$A(l,w,h) = 2(lh + wh + lw) = 6$$

Suppose (l_0, w_0, h_0) is a crical point

where $\nabla V = \lambda \nabla A$ and $A = 6$

By symmetry, (h_0, l_0, w_0) is also a crit pt.

For if we set $\bar{V}(l,w,h) = V(h,l,w)$ and $\bar{A}(l,w,h) = A(h,l,w)$

then (h_0, l_0, w_0) is a crit pt for $(\bar{V}, \bar{A} = 6)$ and

actually $V = \bar{V}$ and $A = \bar{A}$.

So if our problem is simple and there is

only <u>one crit pt</u>, we have $(l_0, w_0, h_0) = (h_0, l_0, w_0)$

$\Rightarrow l_0 = h_0 = w_0 = 1$, i.e. best shape is a cube.

Of course need to check this (easy!) and

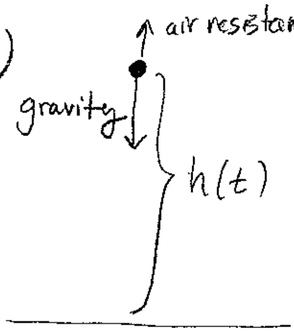
make sure there is a global max...

Partial Differential Equations: (PDE) Section 4.1.

O.D.E. 1) $P(t)$ pop at time t $\quad P'(t) = c\,P(t)$

$$\Rightarrow P(t) = P_0 e^{ct}$$

2)

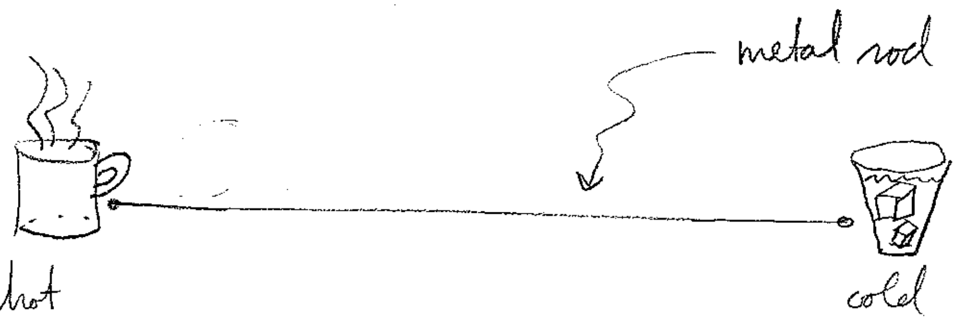


$$h''(t) = -g - a\,h'(t) \Rightarrow$$

$$h(t) = -\frac{1}{a^2}\left(agt + (av_0 + g)\,e^{-at}\right)$$

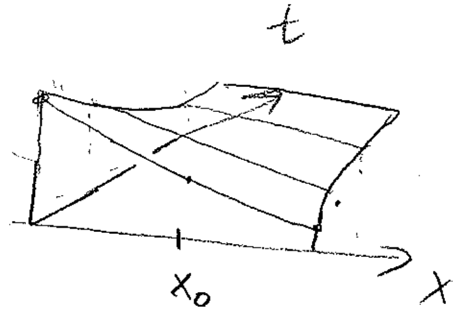
$F = ma$

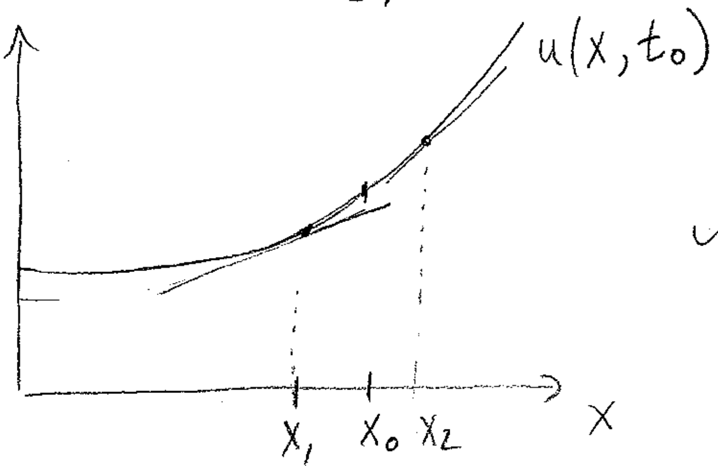
P.D.E.



$u(x,t)$ = temperature at position x at time t.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$





Newton's Law of Cooling:

Flow of heat is proportional to $-\frac{\partial u}{\partial x}$