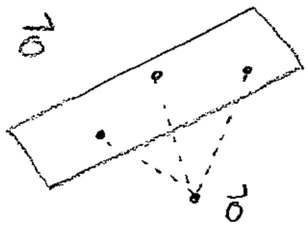
Lecture 21:

Last time: Finding distance from the plane $x - y + 2z = 3$ to $\vec{0}$



i.e. minimizing

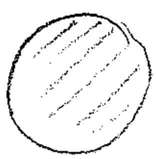
$$f(x,y) = x^2 + y^2 + \frac{1}{4}(3 - x + y)^2$$

Only one critical pt $(x,y) = (1/2, -1/2) \longleftrightarrow (1/2, -1/2, 1)$

No global max.

Closed: D is closed if it "contains all its boundary points."

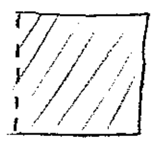
Not closed:



$\|\vec{x}\| \leq 1$

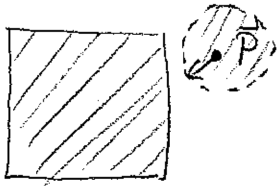
$\|\vec{x}\| < 1$

$0 \leq x \leq 1$
$0 \leq y \leq 1$

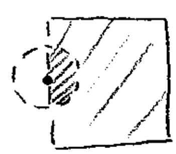


$0 < x \leq 1$
$0 \leq y \leq 1$

Precisely: D is closed if for each point $\vec{p}$ not in D, there is an $r > 0$ such $B(\vec{p}, r)$ misses D



vs.



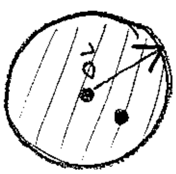
$\vec{p} = (0, 1/2)$

Bounded: means D is contained in some ball:

Extereme Value Thm: D a closed and bounded subset of $\mathbb{R}^n$.
If $f: D \to \mathbb{R}$ is continuous, then f has global min/max on D, which occur either at a) critical points
b) the boundary of D.

Back to the problem at hand. First minimize f on D.

$D = \{ \|\vec{x}\| \leq 2 \}$ in $\mathbb{R}^2$.



Closed, bounded.

One crit pt @ $(1/2, -1/2)$ where $f = 3/2$

On ∂D, $f \geq 4$ so

f has a global min on D of $3/2$. (What about the max? Must occur on ∂D and in fact does so at $(x,y) = (-\sqrt{2}, \sqrt{2})$ where $f \approx 12.5$.)

What about on all of $\mathbb{R}^2$?

Well outside D, $f \geq 4$ so in fact f has a global min at $(1/2, -1/2)$!

<u>Double Check</u>: 2^{nd} derivative test

At $(1/2, -1/2)$, we have

$$H = \begin{pmatrix} 5/2 & 0 \\ 0 & 5/2 \end{pmatrix}$$ which has det > 0 and $f_{xx} > 0 \Rightarrow$ local min.

Another use of critical points: graph sketching

$$f(x,y) = x e^{-(x^2+y^2)}$$

$$\nabla f = \left(e^{-(x^2+y^2)}(1-2x^2), \; -2xy\, e^{-(x^2+y^2)} \right)$$

<u>Crit pts</u>: $\nabla f = \vec{0} \Rightarrow x$ or $y = 0$ by 2^{nd} equation

$$\Rightarrow (x,y) = (\pm 1/\sqrt{2}, 0)$$

$$H = \begin{pmatrix} 2e^{-(x^2+y^2)} x(-3+2x^2) & -2e^{-(x^2+y^2)} y(1-2x^2) \\ -2e^{-(x^2+y^2)} y(1-2x^2) & 2e^{-(x^2+y^2)} x(-1+2y^2) \end{pmatrix}$$

@ $(1/\sqrt{2}, 0)$

$$= \begin{pmatrix} -2\sqrt{2/e} & 0 \\ 0 & -\sqrt{2/e} \end{pmatrix}$$

$\Rightarrow$ local max

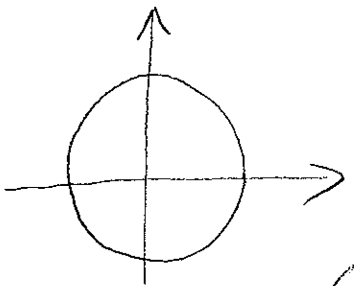
@ $(-1/\sqrt{2}, 0)$

$$= \begin{pmatrix} 2\sqrt{2/e} & 0 \\ 0 & \sqrt{2/e} \end{pmatrix}$$

$\Rightarrow$ local min

Constrained Min / Max.

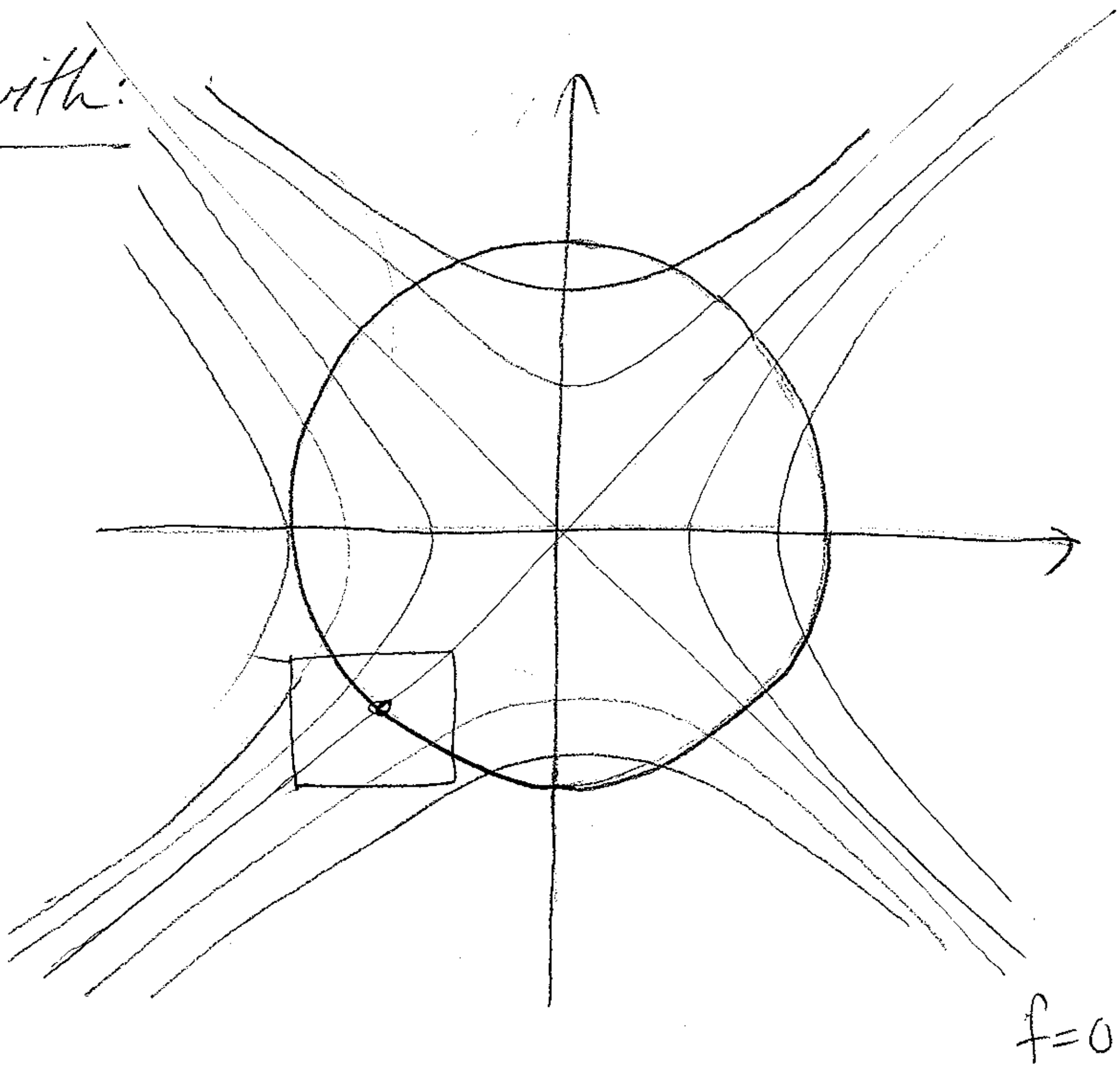
Ex: Find the max of $f(x,y) = x^2 - y^2$ on the unit circle. [Motivate]

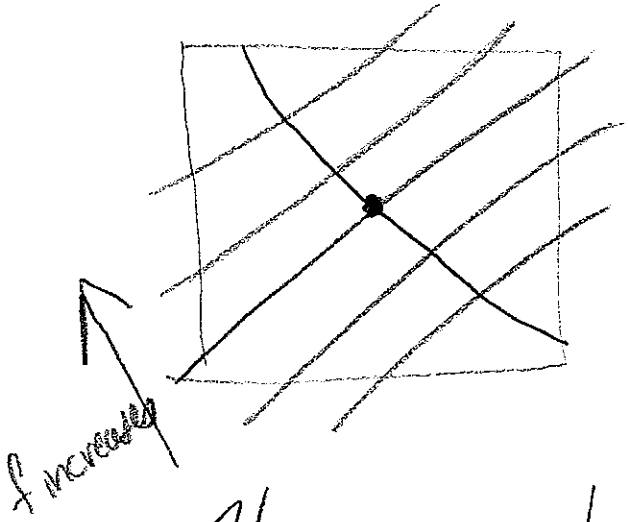


Think of the unit circle as given by $g(x,y) = 1$ where $g(x,y) = x^2 + y^2$

Q: How do we find min/max?

Start with:

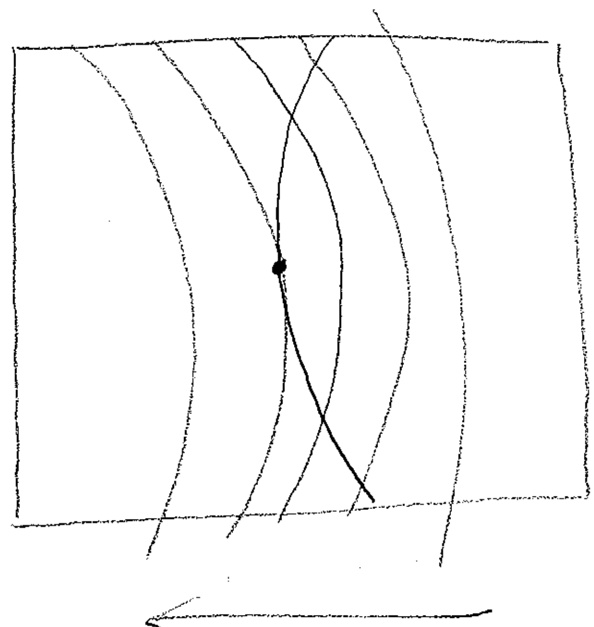




When we have this picture, we don't have a local extrema.

However, when the level sets of f and g are tangent we can have a loc. extrema. In our case, have such tangencies at

$(-1,0), (1,0), (0,1), (0,-1)$

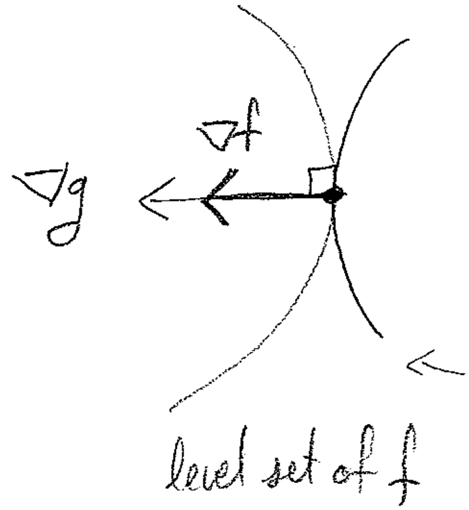


f increases.

$$f = \underbrace{1 \qquad 1}_{\text{global max}} \qquad \underbrace{-1 \qquad -1}_{\text{global min}}$$

← N.B. The circle is closed and bounded.

How can we find these tangencies in general?



Key: gradients point in the same direction, i.e. $\nabla f = \lambda \nabla g$

$\boxed{\text{Lagrange Multipliers}}$

[discovered by Euler!]

Critical Points:

$$g(x,y) = x^2 + y^2 = 1$$

$$\nabla f = (2x, -2y) = \lambda \nabla g = \lambda(2x, 2y)$$

$$\Leftrightarrow \quad 2x = \lambda 2x \quad \text{and} \quad -2y = \lambda 2y$$

If $x \neq 0$ then $\lambda = 1 \Rightarrow y = 0 \Rightarrow x = \pm 1$

If $y \neq 0$ then $\lambda = -1 \Rightarrow x = 0 \Rightarrow y = \pm 1$.

So critical points are

	$(1,0)$	$(-1,0)$	$(0,1)$	$(0,-1)$
λ	1	1	-1	-1
f	1	1	-1	-1
	global max		global min	