Lecture 20: Min/max in action

Last time: $f: \mathbb{R}^2 \to \mathbb{R}$ with a critical point at $\vec{x}_0$

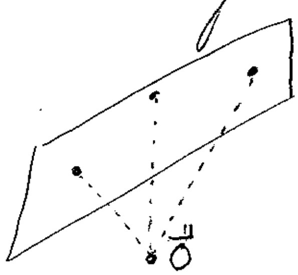
$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(\vec{x}_0) & \frac{\partial^2 f}{\partial y \partial x}(\vec{x}_0) \\ \frac{\partial^2 f}{\partial x \partial y}(\vec{x}_0) & \frac{\partial^2 f}{\partial y^2}(\vec{x}_0) \end{pmatrix} \qquad D = \det H$$

a) $D > 0$, $f_{xx}(\vec{x}_0) > 0 \Rightarrow$ local min

b) $D > 0$, $f_{xx}(\vec{x}_0) < 0 \Rightarrow$ local max

c) $D < 0 \Rightarrow$ saddle

Ex: Find distance from the plane $x - y + 2z = 3$ to $\vec{0}$.



$$z = \frac{3 - x + y}{2}$$

Want to minimize $f(x,y) = \begin{pmatrix} \text{dist from} \\ (x, y, \frac{3-x+y}{2}) \\ \text{to } \vec{0} \end{pmatrix}^2 = x^2 + y^2 + \frac{1}{4}(3 - x + y)^2$

[The square is just to make the computations easier.]

Critical Points: $\nabla f = \vec{0}$ (or undefined)

$$\frac{\partial f}{\partial x} = 2x - \frac{1}{2}(3 - x + y) = \frac{5}{2}x - \frac{1}{2}y - \frac{3}{2}$$

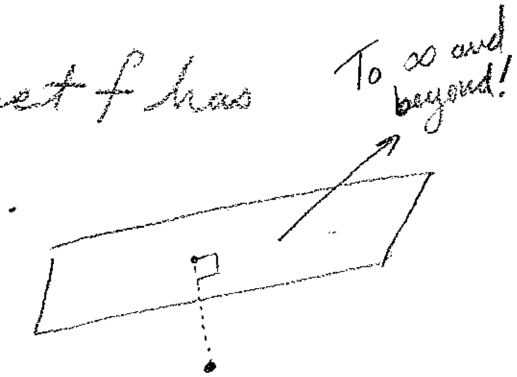
$$\frac{\partial f}{\partial y} = 2y + \frac{1}{2}(3 - x + y) = -\frac{1}{2}x + \frac{5}{2}y + \frac{3}{2}$$

Solve $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \iff \begin{matrix} 5x - y = 3 \\ -x + 5y = -3 \end{matrix} \iff \begin{matrix} x = 1/2 \\ y = -1/2 \end{matrix}$

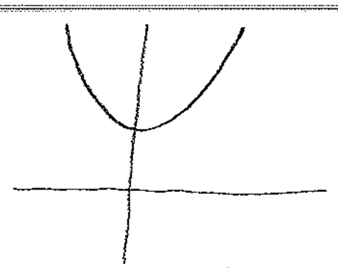
Since there is only one critical point this must be our minimum, and so the closest point is $(1/2, -1/2, 1)$ at distance $\sqrt{3/2} \approx 1.2$.

Hey, wouldn't the same reasoning "show" that the <u>maximum</u> distance is also achieved at $(1/2, -1/2, 1)$?

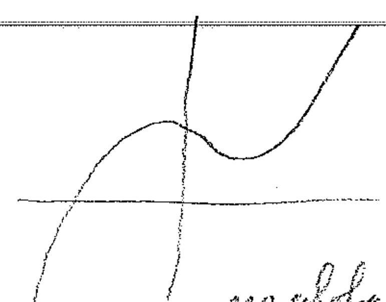
True, we're missing something. In fact f has no maximum, as is clear geometrically. From the picture, it looks like we have a min but clearly we need to think things through more carefully.



<u>One var</u>: May or may not have <u>global</u> extreme values

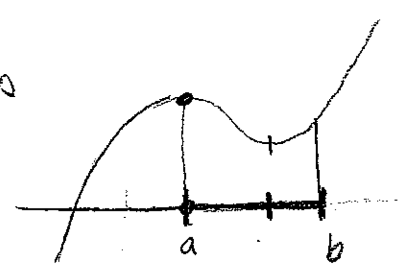


no global max



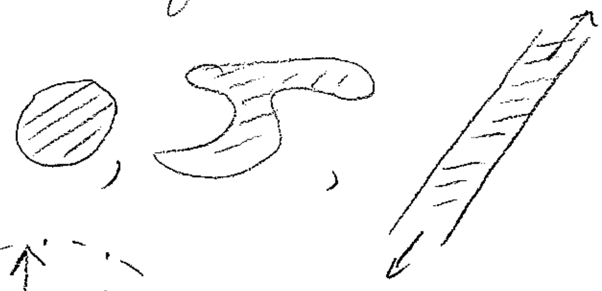
no global max or min.

<u>Extreme Value Theorem</u>: f continuous function on $[a,b] = \{a \le x \le b\}$. Then f has a global min and max.

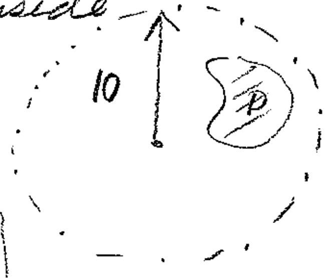


Addendum: These global min/max occur at either a) a critical pt b) one of the end pts $\{a, b\}$.

Multi Var: D a subset of $\mathbb{R}^2$

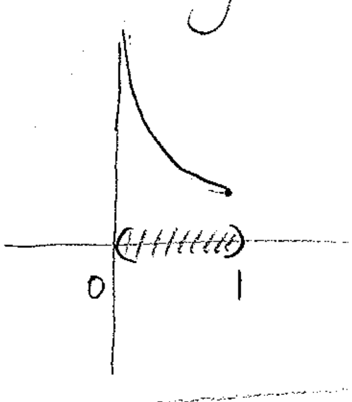


Bounded: D is contained inside some ball.



$B_{10}(\vec{0})$

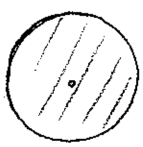
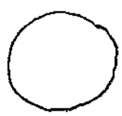
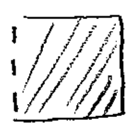
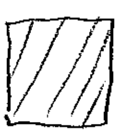
Not enough in 1-var, e.g.



$(0,1) = \{0 < x < 1\}$

$f(x) = 1/x$.

Closed: D is closed if it "contains all its boundary points."

Closed	Not closed
$\{\lVert\vec{x}\rVert \le 1\}$ 	$\{\lVert\vec{x}\rVert < 1\}$ 
 $\{\lVert\vec{x}\rVert = 1\}$	 $\left\{ (x,y) \text{ where } \begin{array}{l} 0 < x \le 1 \\ 0 \le y \le 1 \end{array} \right\}$
 $\left\{ (x,y) \text{ where } \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1 \end{array} \right\}$	

<u>Formally</u>: D is closed if for each point $\vec{p}$ not in D, there is an $r > 0$ such that $B_r(\vec{p})$ miss D

$\vec{p} = (0, 1/2)$

<u>Extreme value theorem</u>: Suppose D is a closed and bounded subset of $\mathbb{R}^n$. If $f: D \to \mathbb{R}$ is continuous, then f has global extrema on D, which occur either at a) critical point

b) the boundary of D.

Back to problem at hand.

$D = \{\|\vec{x}\| \leq 2\}$

On D there's one crit pt $(1/2, -1/2)$ where $f = 3/2$

On ∂D, $f \geq 4$. So is the global min of f on D.

Outside of D, $f \geq 4$ so in fact f has a global min on $\mathbb{R}^2$ of $3/2$, acquired at $(1/2, -1/2)$. <u>Double check</u>: Apply 2^{nd} der. test