Lecture 19: Taylor series for min/max (§4.2)

Last time: $f:\mathbb{R}^2 \to \mathbb{R}$ has mins/maxes at

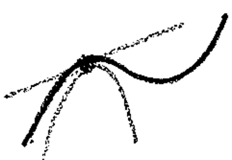
critical points: $\nabla f(\vec{x}_0) = \left(\frac{\partial f}{\partial x}(\vec{x}_0), \frac{\partial f}{\partial y}(\vec{x}_0)\right) = \vec{0}$

min   max   saddle

x^2+y^2 $-x^2-y^2$ x^2-y^2

Next time: §4.3   HW: Handout/web.

One var: $f(x_0+h) = f(x_0) + f'(x_0)h + E(h)$ where $\lim_{h\to 0} \frac{E(h)}{h} = 0$



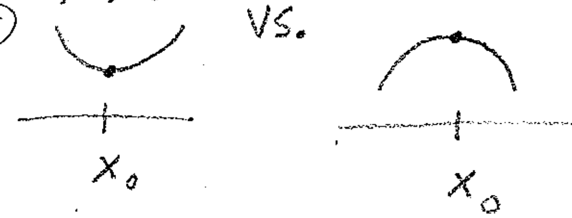
$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + E(h)$$

where $\lim_{h\to 0} \frac{E(h)}{h^2} = 0$

If $f'(x_0) = 0$, then

$$f(x_0+h) \approx f(x_0) + \frac{f''(x_0)}{2}h^2 \implies$$

$f''>0$ vs. $f''<0$



[so this is one way to think about the 2nd derivative test.]

Multivar: $f:\mathbb{R}^2 \to \mathbb{R}$

Diff at $\vec{x}_0$: $f(\vec{x}_0+\vec{h}) \approx f(\vec{x}_0) + Df(\vec{x}_0)\vec{h}$

$$\iff f(x_0+h, y_0+k) \approx f(\vec{x}_0) + \frac{\partial f}{\partial x}(\vec{x}_0)h + \frac{\partial f}{\partial y}(\vec{x}_0)k$$

[One way of thinking about Taylor series is approx by polynomials, the simplest of functions.]

Next level:

$$f(x_0+h,\ y_0+k) = f(\vec{x}_0) + \frac{\partial f}{\partial x}(\vec{x}_0)\,h + \frac{\partial f}{\partial y}(\vec{x}_0)\,k + ah^2 + bhk + ck^2$$

[Have class figure out what a, b and c are.] $+ E(h,k)$ ← smaller than other terms.

A: $a = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\vec{x}_0)$, $b = \frac{\partial^2 f}{\partial x \partial y}(\vec{x}_0)$ $\left(\text{or } \frac{\partial^2 f}{\partial y \partial x}(\vec{x}_0)\right)$, $c = \frac{\partial^2 f}{\partial y^2}(\vec{x}_0)$

Ex: $f(x,y) = \sin\left(\sqrt{1+\frac{x^2}{2+\cos y}} - e^{-xy}\right)$

$\vec{0}$ is a critical point so we need to decide what the graph looks like near $\vec{0}$.

$$f(x,y) = \frac{1}{6}x^2 + xy + \underbrace{E(x,y)}_{\text{only terms of degree 3 or more.}}$$

Q: Does $\frac{1}{6}x^2 + xy$ have a min/max at $\vec{0}$?

A: No, it's a saddle.

Hessian:

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(\vec{x}_0) & \frac{\partial^2 f}{\partial x \partial y}(\vec{x}_0) \\ \frac{\partial^2 f}{\partial y \partial x}(\vec{x}_0) & \frac{\partial^2 f}{\partial y^2}(\vec{x}_0) \end{pmatrix}$$

Then

means as a row vector.

$$f(\vec{x}_0+\vec{h}) = f(\vec{x}_0) + [Df(\vec{x}_0)]\vec{h} + \frac{1}{2}\vec{h}^T H \vec{h} + E(\vec{h})$$

works for any [sufficiently differentiable] function $f: \mathbb{R}^n \to \mathbb{R}$, e.g. for $f: \mathbb{R}^3 \to \mathbb{R}$ have

$$H = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}$$

2^{nd} derivative test:

Suppose $\vec{x}_0$ is a critical point of $f: \mathbb{R}^2 \to \mathbb{R}$.

Let $D = \det(H(\vec{x}_0))$ then

a) If $D > 0$ and $f_{xx}(\vec{x}_0) > 0 \Longrightarrow$ local min

b) If $D > 0$ and $f_{xx}(\vec{x}_0) < 0 \Longrightarrow$ local max

c) If $D < 0 \Longrightarrow$ neither, a saddle.

Other cases, e.g. $D = 0$ are inconclusive.

min	max	saddle
Ex: x^2+y^2 at $\vec{0}$	$-x^2-y^2$ at $\vec{0}$	x^2-y^2 at $\vec{0}$
$H = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$H = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$	$H = \begin{pmatrix} +2 & 0 \\ 0 & -2 \end{pmatrix}$
$D = 4$	$D = 4$	$D = -4$
$f_{xx} > 0$	$f_{xx} < 0$	

What's going on "under the hood"?

Simple case: $f(x_0+h, y_0+k) = ah^2 + bk^2 + E(h,k)$

Important Fact: Can rotate the coordinates so that f has this form. [Ex on HW.] Such a rotation does not change $\det H$.

A similar test is available for any $f: \mathbb{R}^n \to \mathbb{R}$ but need to talk about eigenvalues of the Hessian matrix; you'll see this in linear algebra...