Lecture 17: Higher Order Partial Derivatives and Min/Max

Reminder: Email me about what you want to see in the review on Wed.

HW: None

Note: Starting Section 4 today [Not on the exam.]

Higher order partials:

One var: $\frac{df}{dx}$, $\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$, $\frac{d^3f}{dx^3}$

mixed partials

Multi: $f: \mathbb{R}^2 \to \mathbb{R}$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}$$

[Query: What is this?
A: $\mathbb{R}^2 \to \mathbb{R}$]

Ex: $f(x,y) = x\sin(xy^2)$

$$\frac{\partial f}{\partial x} = \sin(xy^2) + xy^2\cos(xy^2) \qquad \frac{\partial f}{\partial y} = x\cos(xy^2)2xy$$
$$= 2x^2y\cos(xy^2$$

Point both $\frac{\partial f}{\partial y\partial x}$ and $\frac{\partial f}{\partial x \partial y}$ are the same and are

$$4xy\cos(xy^2) - 2x^2y^3\sin(xy^2)$$

Thm: If $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are defined and continuous near $\vec{x}_0$ then $\frac{\partial^2 f}{\partial x \partial y}(\vec{x}_0) = \frac{\partial^2 f}{\partial y \partial x}(\vec{x}_0)$

[Very useful, both theoretically and practically.]

Reason: Both are

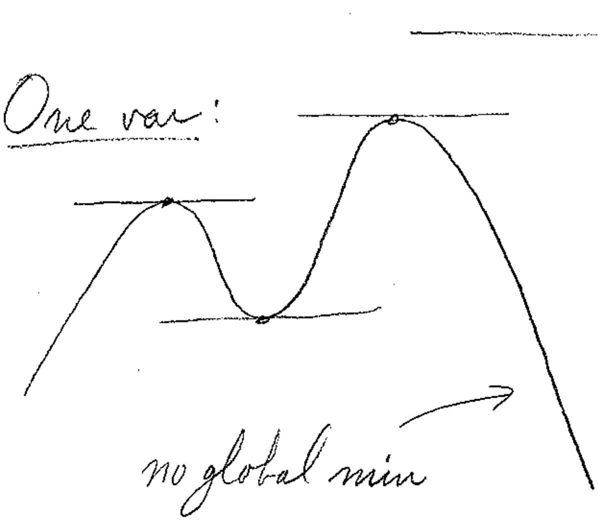
$$\lim_{(h_1,h_2)\to(0,0)} \frac{f(x_0+h_1, y_0+h_2) - f(x_0+h_1, y_0) - f(x_0, y_0+h_2) + f(x_0, y_0)}{h_1 h_2}$$

See text for details

Survey results:

1) Many feel HW grading is harsh.
2) Many are concerned about the exam / grade in course.
3) Many wish for more "HW-like" examples in class.

One var:



Min / Max occur at critical points where $f' = 0$

Local vs. Global Extrema

2^{nd} der test:

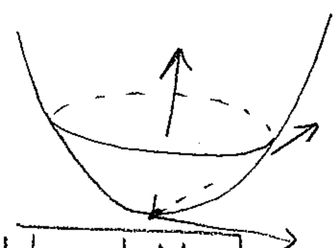
$f'' < 0$ vs $f'' > 0$.

<u>Goal</u>: Do the same for $f: \mathbb{R}^n \to \mathbb{R}$ [Explain]

<u>Critical Points</u>: $\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right) = \vec{0}$

Every local extreme is a critical point. [as discussed yesterday]

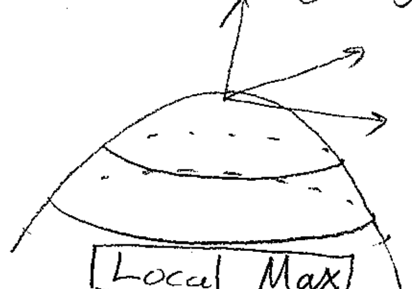
The hard bit is the analog of the 2^{nd} der test.



$\boxed{\text{Local Min}}$

$f(x,y) = x^2 + y^2$

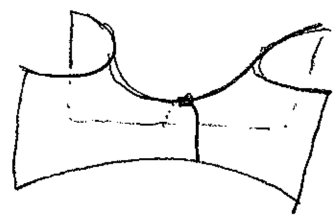
$\nabla f = (2x, 2y)$



$\boxed{\text{Local Max}}$

$f(x,y) = -x^2 - y^2$

$\nabla f = (-2x, -2y)$



$\boxed{\text{Neither: Saddle}}$

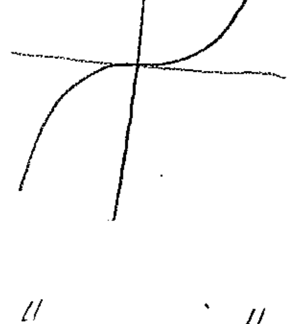
$f(x,y) = x^2 - y^2$

$\nabla f = (2x, -2y)$

<u>Note</u>: $\vec{0}$ is a critical point for all 3.

In one var, there was also a "neither" case,

e.g. $f(x) = x^3$



$f'(0) = 0$

$f''(0) = 0$

but this is not "generic", whereas a saddle is "stable."

Where did the 2^{nd} der test come from?

One point of view: Taylor series

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \underline{\underline{E(h)}}$$

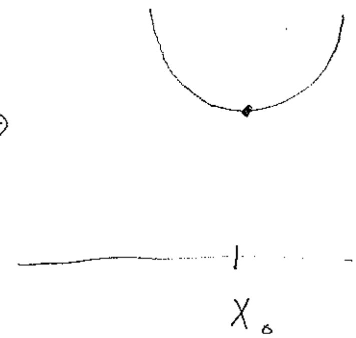
very small

$$\lim_{h\to 0} \frac{E(h)}{h^2} = 0$$

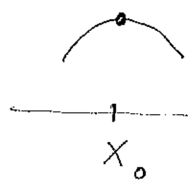
At a critical point have

$$f(x_0+h) = f(x_0) + \frac{f''(x_0)}{2}h^2 + E(h)$$

If $f''(x_0) > 0$ graph looks like $\longrightarrow$



If $f''(x_0) < 0$ graph looks like



<u>Q1</u>: What should Taylor series be for functions of several vars?

<u>Q2</u>: What are the possible local pictures?