Lecture 16:

Last time: $f: \mathbb{R}^2 \to \mathbb{R}$

Gradient $\nabla f(\vec{x}_0) = \left(\frac{\partial f}{\partial x}(\vec{x}_0), \frac{\partial f}{\partial y}(\vec{x}_0)\right)$

points in the direction of greatest increase of f

HW: No additional

Reminder: Exam in class this Thursday. Extra Office Hours, etc

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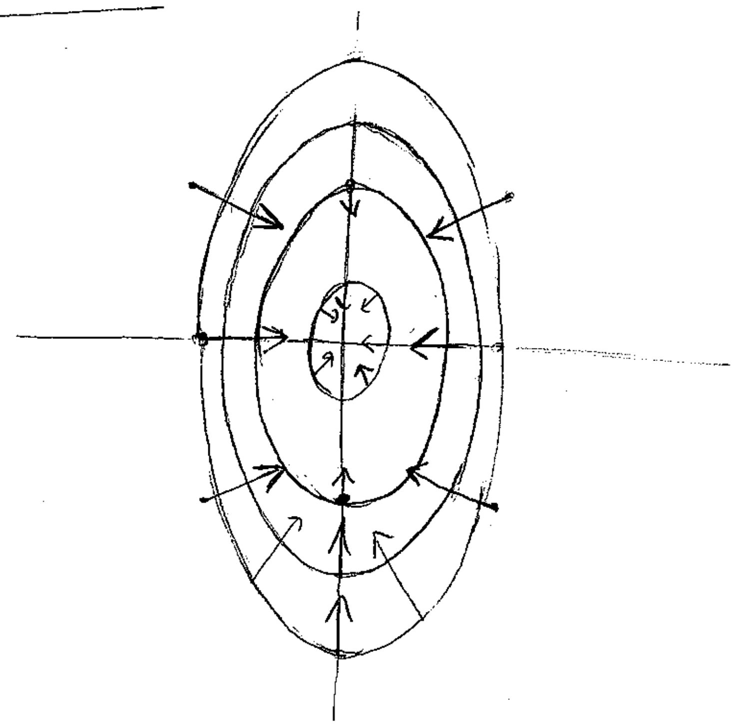
Ex: $f(x,y) = 1 - 4x^2 - y^2$

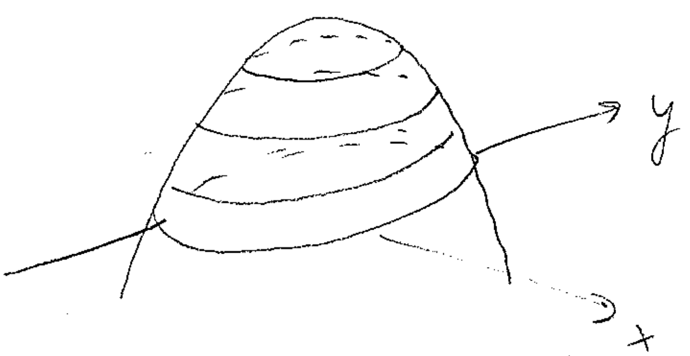
$\nabla f = (-8x, -2y)$



What is the level set

$f = 0$: $4x^2 + y^2 = 1$

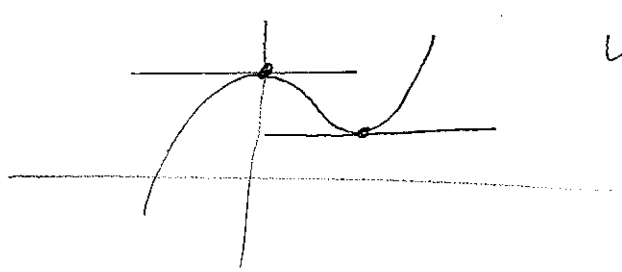
$f = -3$: $4x^2 + y^2 = 4$



What is $\nabla f(\vec{0})$?

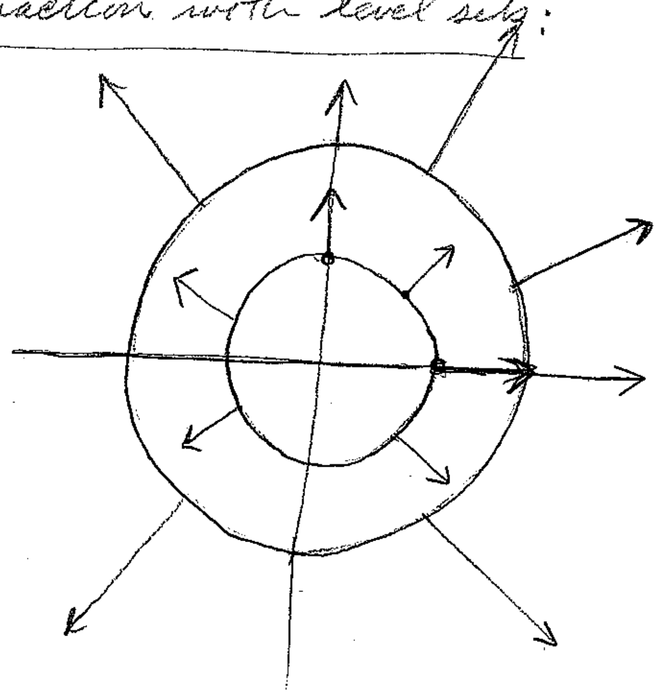
Ans: $\vec{0}$.

Recall:



A min/max can only occur when $\nabla f = \vec{0}$

Interaction with level sets:

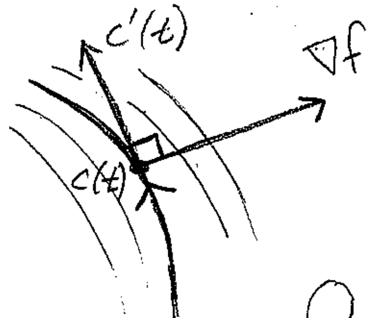


$$f(x,y) = x^2 + y^2$$

$$\nabla f = (2x, 2y)$$

Notice: ∇f is perpendicular to each level curve

Why?



Consider a path $c : \mathbb{R} \to \mathbb{R}^2$ in a level set

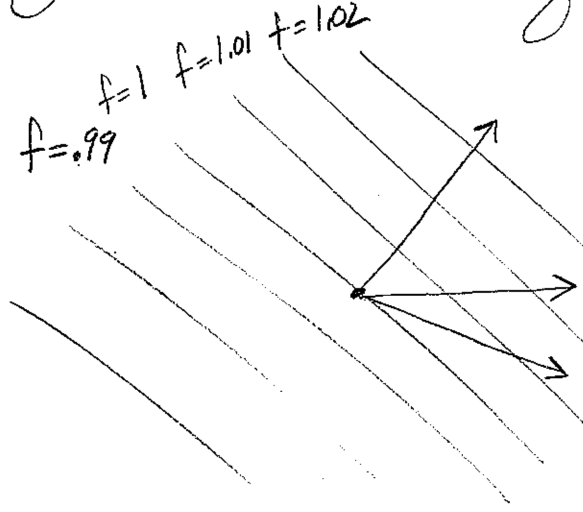
$$0 = \frac{d}{dt} f \circ c = Df(c(t))\, c'(t)$$

$$= \nabla f(c(t)) \cdot c'(t) \implies c' \text{ and } \nabla f$$

are at right angles.

Annother way of thinking about:

Locally, the level sets of f look like those of a linear function (typically).



Then if we want to increase f as much as possible, we should head at right angles to the level sets.

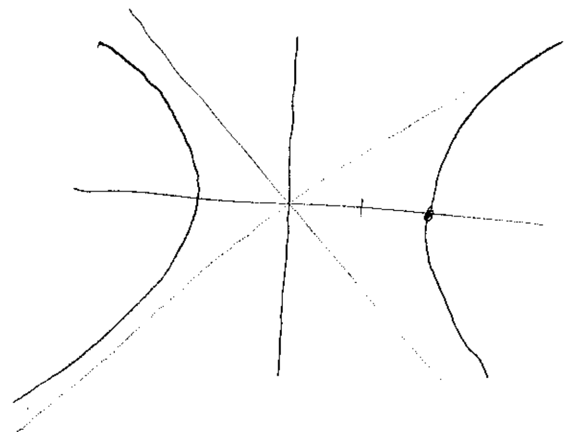
The gradient makes sense for any $f: \mathbb{R}^n \to \mathbb{R}$

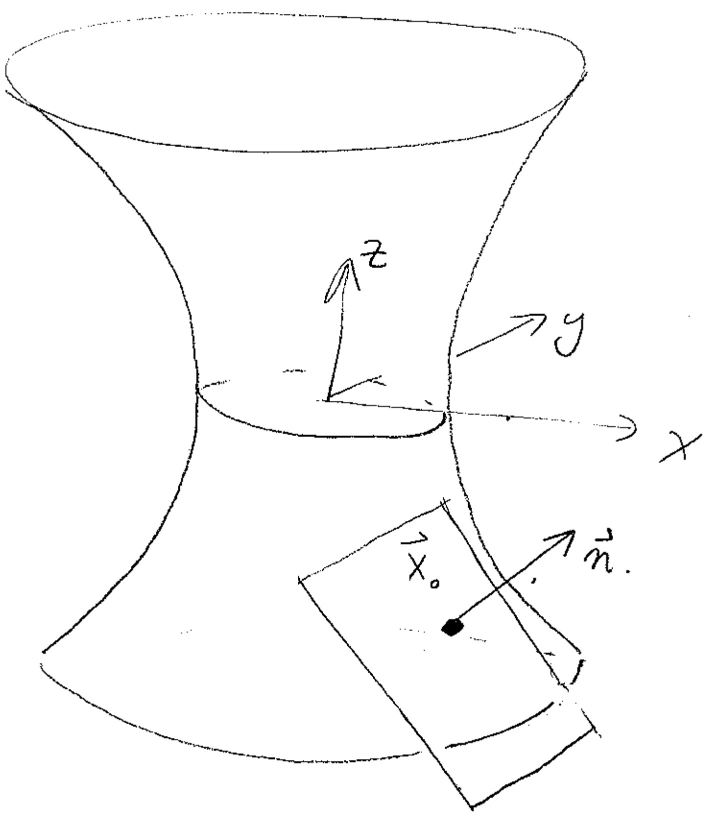
E.g. $f: \mathbb{R}^3 \to \mathbb{R}$, $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

Has the properties discussed [1) Points in dir of greatest increase 2) vanishes at min/max 3) $\perp$ to level sets.]

<u>Ex</u>: $f(x,y,z) = x^2+y^2-z^2$

$\{f = 4\}$





$\vec{x}_0 = (3, 2, -3)$

Find tangent plane at $\vec{x}_0$.

What do we need?

a normal vector.

$$\nabla f = (2x, 2y, -2z)$$

$$\vec{n} = \nabla f(\vec{x}_0) = (6, 4, -6)$$

Now the plane is given by

$$(\vec{x} - \vec{x}_0) \cdot \vec{n} = 0$$

$$(x-3, y-2, z-3) \cdot (6, 4, -6) = 0$$

$$\boxed{6x + 4y - 6z = 8}$$