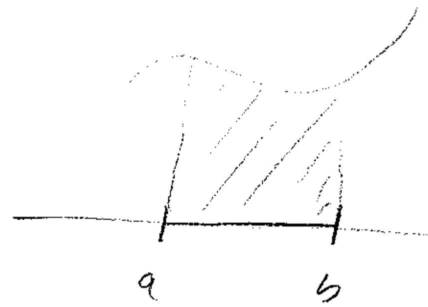
Lecture 44: Midterm Review

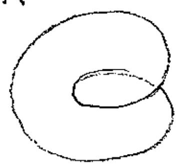
Integrals we have known

1-dimension:

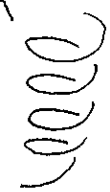
Basic: $\int_a^b f(x)\,dx$



1-d in $\mathbb{R}^n$: $\mathbb{R}^2$:



$\mathbb{R}^3$



Curves

Path integrals:

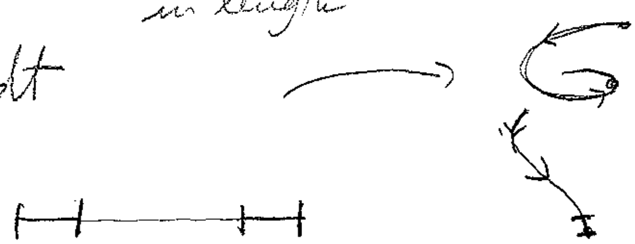
$f: \mathbb{R}^n \to \mathbb{R}$ a function $\boxed{\int_C f\,ds}$ ← from which the others are derived.

$\vec{F}$ a vector field on $\mathbb{R}^n$ $\int_C \vec{F} \cdot ds$

$\vec{F}$ a vector field on $\underline{\mathbb{R}^2}$: $\int_C (\vec{F} \cdot \vec{n})\,ds$

To compute, need a parameterization $c: [a,b] \to \mathbb{R}^n$ of our curve

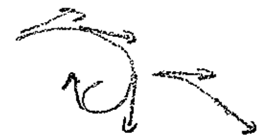
$$\int_C f\,ds = \int_a^b f(c(t)) \underbrace{\|c'(t)\|}_{\text{accounts for change in length}} dt$$



$$\int F \cdot ds = \begin{pmatrix} \text{circulation:} \\ \text{the tangential} \\ \text{part of } \vec{F} \end{pmatrix} = \int_a^b \vec{F}(c(t)) \cdot c'(t)\, dt$$

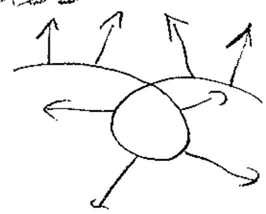
$$= \int_a^b \vec{F}(c(t)) \cdot \frac{c'(t)}{\|c'(t)\|} \|c'(t)\|\, dt = \int_C (\vec{F} \cdot \vec{u})\, ds$$

where u is the <u>unit</u> tangent vector field to C



In $\mathbb{R}^2$, C also has two (unit) normal vector fields

so also have $\int_C (\vec{F} \cdot \vec{n})\, ds$



[Path integral vs. Line integral: discuss terminology.]

<u>2-dimensional</u>:

<u>Basic</u>: D region in $\mathbb{R}^2$, $f: D \to \mathbb{R}$

$$\iint_D f\, dA = \iint_D f\, dx\, dy$$

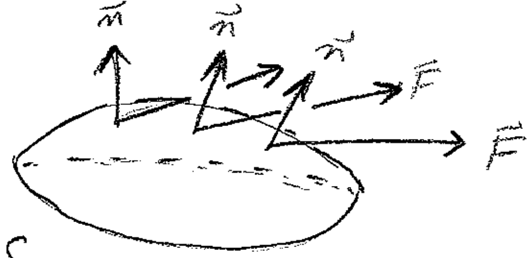
$$\iint_D 1\, dA = \text{Area}(D)$$

<u>2^d in $\mathbb{R}^3$</u>: Surfaces

- $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$\iint_S f\, dA = \iint_S f\, dS$$ ← alternate notation

- $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ vector field



$$\iint_S (\vec{F} \cdot \vec{n})\, dA = \iint_S \vec{F} \cdot dS$$

[Point out difference between $\cdot ds$ and $\cdot dS$.]

$r : (D \text{ in } \mathbb{R}^2) \longrightarrow (S \text{ in } \mathbb{R}^3)$ parametrization

$$\iint_S f\, dA = \iint_D f(r(u,v))\, \|T_u \times T_v\|\, du\, dv$$

$$\iint_S (\vec{F} \cdot \vec{n})\, dA = \iint_S \left(\vec{F}(r(u,v)) \cdot \vec{n}(u,v) \right) \|T_u \times T_v\|\, du\, dv$$

$$= \iint_S \vec{F}(r(u,v)) \cdot \frac{T_u \times T_v}{\|T_u \times T_v\|} \|T_u \times T_v\|\, du\, dv = \iint_S \vec{F}(r(u,v)) \cdot (T_u \times T_v)\, du\, dv$$

<u>3-dimensional</u>:

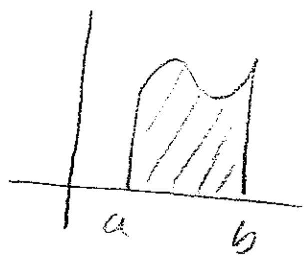
<u>Basic</u>: R region in $\mathbb{R}^3$, $f : \mathbb{R}^3 \to R$

$$\iiint_R f\, dV = \iiint_R f\, dx\, dy\, dz$$

$$\iiint_R 1\, dV = \text{Volume}(R)$$

Comments:

1) These notions are interrelated: the area of



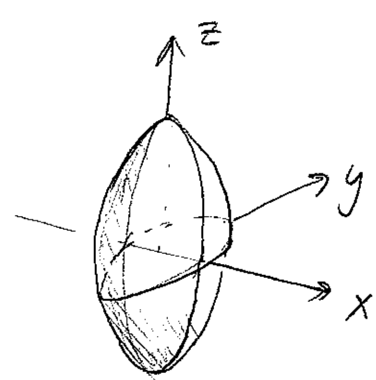
can be found by either $\int_a^b f\,dx$ or $\iint_D 1\,dA$.

2) Change of coordinates in $\mathbb{R}^2$ is really a special case of doing a surface integral where our parameterization is to a region in $\mathbb{R}^2$ instead of a surface in $\mathbb{R}^3$.

$$\iint_{D^*} f\circ T(u,v)\,|\det DT|\,du\,dv = \iint_D f\,dA$$

Examples of Parametrizations:

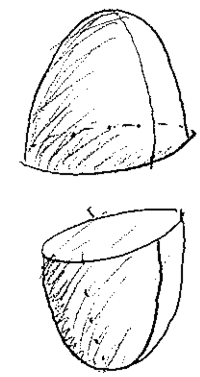
$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$$



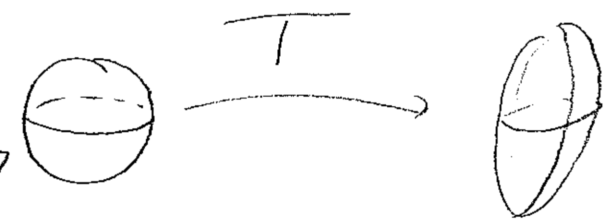
Method 1: Split into two pieces

$$r(u,v) = \left(u, v, 3\sqrt{1 - x^2 - y^2/4}\right)$$

$$s(u,v) = \left(u, v, -3\sqrt{1 - x^2 - \tfrac{1}{4}y^2}\right.$$

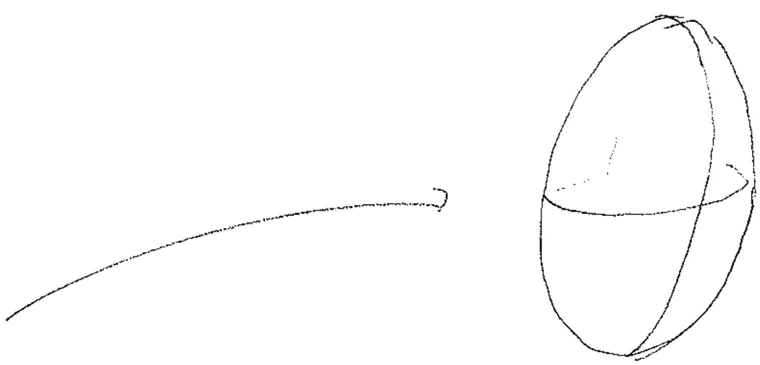


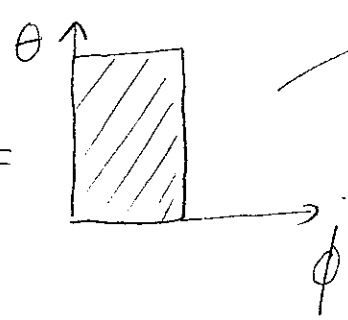
Method 2: "Elliptical coordinates"



$$T(x,y,z) = (x, 2y, 3z)$$

$$x = \sin\phi\cos\theta$$
$$y = \sin\phi\sin\theta$$
$$z = \cos\phi$$



D = 

$$r(\phi,\theta) = (\sin\phi\cos\theta, 2\sin\phi\sin\theta, 3\cos\phi)$$