Lecture 43: More applications of the Divergence Thm.

HW: Review problems on web

Review: Email me suggestions

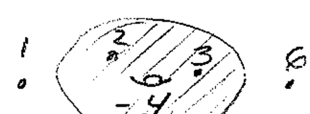
Charges Q_i at locations $\vec{x}_i$

Electric Field $= \vec{E}(\vec{r}) = \sum \vec{E}_i(\vec{r})$ where $\vec{E}_i = \dfrac{Q_i}{4\pi\varepsilon_0} \dfrac{(\vec{r} - \vec{x}_i)}{\|\vec{r} - \vec{x}_i\|^3}$

Gauss's Law: R a region in $\mathbb{R}^3$. Then

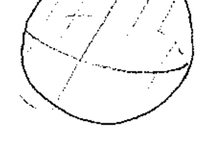
$$\iint_{\partial R} (\vec{E}(\vec{r}) \cdot \vec{n})\, dA = \frac{1}{\varepsilon_0} \begin{pmatrix} \text{total charge} \\ \text{inside } R \end{pmatrix}$$

[Mention uses.]



While charge is actually quantized, it often makes sense to consider "charge densities."

$$\rho(x,y,z) \quad \left(\text{has units } \frac{(\text{charge})}{(\text{volume})}\right)$$



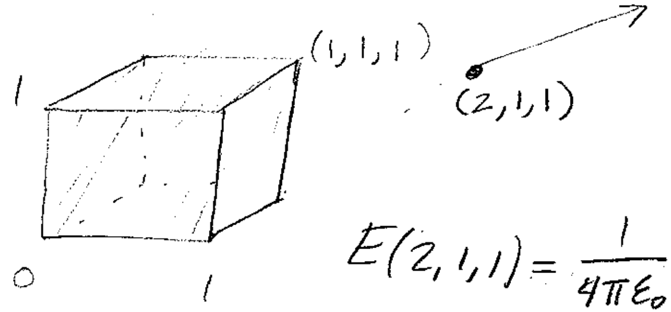
In this case

$$E(\vec{r}) = \iiint_R \frac{1}{4\pi\varepsilon_0} \frac{\rho(\vec{x})(\vec{r} - \vec{x})}{\|\vec{r} - \vec{x}\|^3}\, dx\,dy\,dz$$

where we are integrating over $\vec{x} = (x,y,z)$ and now the integrand is vector-valued.

$$\rho(x,y,z) = x$$

Ex:

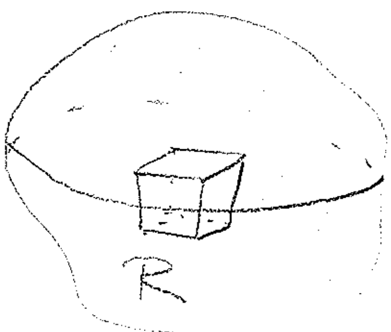


$$E(2,1,1) = \frac{1}{4\pi\varepsilon_0}\int_0^1\int_0^1\int_0^1 \frac{x\,(2-x,\,1-y,\,1-z)}{\left((2-x)^2+(1-y)^2+(1-z)^2\right)^{3/2}}\,dx\,dy\,dz$$

$$\approx \frac{1}{4\pi\varepsilon_0}\,(0.19,\ 0.07,\ 0.07)$$

On the other hand, Gauss's Law says that if R is a region containing this cube, then

$$\iint_{\partial R} (\vec{E}\cdot\vec{n})\,dA = \frac{1}{\varepsilon_0}\iiint \rho(x,y,z)\,dV = \frac{1}{2\varepsilon_0}$$



Another advantage of looking at a charge density is that $\vec{E}$ is now defined everywhere [Note to self: This is because $\frac{1}{r^2}$ is integrable near $\vec{0}$.]

Thus the divergence theorem applies and

$$\iiint_R \operatorname{div}\vec{E}\,dV = \iint_{\partial R} (\vec{E}\cdot\vec{n})\,dA = \frac{1}{\varepsilon_0}\iiint_R \rho\,dV$$

for every region R. Therefore [Discuss 1^d example]

$$\operatorname{div}\vec{E} = \frac{1}{\varepsilon_0}\rho.$$

at each point. This is Maxwell's 1st Eqn.

Heat Flow: $u(x,y,z,t)$ = temperature at (x,y,z) at time t

$$\frac{\partial u}{\partial t} = c\,\Delta u = c\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

Newton's Law of Cooling:

Heat flow $= -k\ \operatorname{grad}(u)$ [here t is fixed]

The rate heat flows into a region R is

$$\iint_{\substack{\partial R\\ \text{inward normal}}} -k\operatorname{grad}(u)\,dA = \iiint_R k\operatorname{div}(\operatorname{grad} u)\,dV$$

$$= \iiint_R K \Delta u \, dV.$$

The amount of heat energy in R is given by

$$\iiint_R \sigma \rho u \, dV \quad \text{where} \quad \begin{array}{l} \sigma = \text{specific heat} \\ \rho = \text{mass density} \end{array}$$

are constants.

and the rate it is changing is

$$\frac{\partial}{\partial t} \iiint_R \sigma \rho \, u(x,y,z,t) \, dx dy dz$$

$$= \iiint_R \sigma \rho \frac{\partial u}{\partial t} \, dV$$

Thus as this is true for all regions,

$$\boxed{\frac{\partial u}{\partial t} = \frac{k}{\sigma \rho} \Delta u}$$

[Units: Heat is energy J, so heat flux is W, K is in W/mK
ρ in g/m^3, σ in J/gK.]