Lecture 42: Applications of the Divergence Theorem

HW: Web. Next Time:

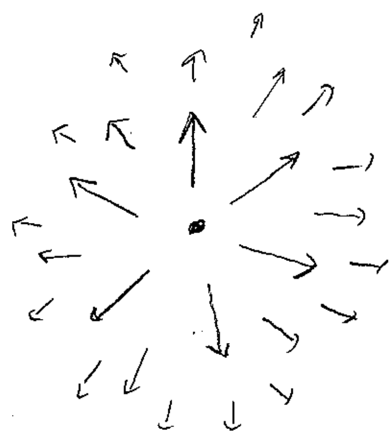
Reminder: Exam Thursday!

Divergence Theorem: D a region in $\mathbb{R}^3$, $\vec{n}$ outward unit normal vector field on ∂D. $\vec{F}: D \to \mathbb{R}^3$ a vector field. Then

$$\iint_{\partial D} (\vec{F} \cdot \vec{n})\, dA = \iiint_D \operatorname{div} \vec{F}\, dV$$

[Recall geometric meaning thereof.]

Electrostatics: Particle of charge Q at $\vec{0}$

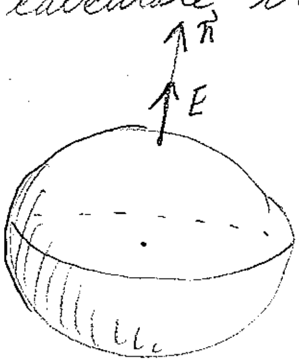


Electric Field at $\vec{r} = (x, y, z)$

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{\|\vec{r}\|^3} \vec{r} \longleftarrow \left[\text{inverse-square law}\right]$$

If another particle is at position $\vec{r}$ with charge P, then it experiences a force $\vec{F} = P\vec{E}(\vec{r})$.

Let's calculate the flux of $\vec{E}$ on a sphere of radius ρ_0 about $\vec{0}$.



$$\iint_{S_{\rho_0}} (\vec{E} \cdot \vec{n})\, dA = \iint \frac{Q}{4\pi\varepsilon_0 \rho_0^2}\, dA$$

$$= \frac{Q}{\varepsilon_0}$$

Now let R be the region bounded by S_{ρ_0} and try to calculate:

$$\iiint_R \operatorname{div} E \, dV.$$ We have: $\vec{E} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{x}{\rho^3}, \frac{y}{\rho^3}, \frac{z}{\rho^3}\right)$

where $\rho = \|\vec{r}\| = \sqrt{x^2+y^2+z^2}$

Noting $\frac{\partial \rho}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x) = \frac{x}{\rho}$, we easily calculate

$$\operatorname{div}\vec{E} = \frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{\rho^3 - 3\rho x^2}{\rho^6} + \frac{\rho^3 - 3\rho y^2}{\rho^6} + \frac{\rho^3 - 3\rho z^2}{\rho^6}\right)$$

$$= 0$$

So $\iiint_R \operatorname{div} E \, dV = 0$, violating the Divergence Theorem?!

What's gone wrong here? <u>Ans</u>: E not defined on all of D since $\frac{1}{\rho^3}\vec{r}$ makes no sense at $\vec{0}$.

<u>Gauss's Law</u>: Let R be a region in $\mathbb{R}^3$.

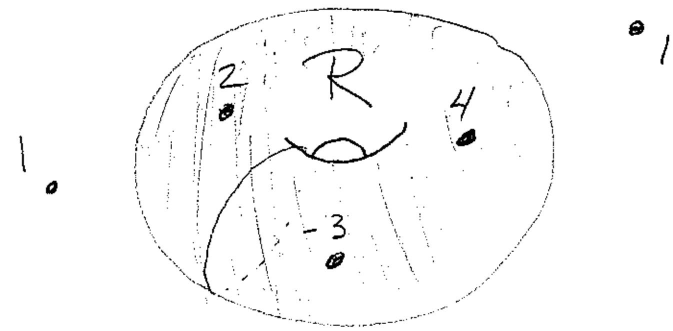
$\vec{E}$ the electric field resulting from charges Q_i at positions $\vec{p}_i$. Then

$$\iint_{\partial R} (\vec{E}\cdot\vec{n})\, dA = \frac{1}{\varepsilon_0}\left(\sum_{\substack{i \text{ where } p_i \\ \text{is in } R}} Q_i\right)$$

Here

$$\vec{E}(\vec{r}) = \sum \vec{E}_i(\vec{r}) = \sum \overbrace{\frac{Q_i}{4\pi\varepsilon_0} \frac{1}{\|\vec{r}-\vec{p}_i\|^3} (\vec{r}-\vec{p}_i)}^{E_i}$$

Ex:



$$\iint_{\partial R} (\vec{E}\cdot\vec{n})\,dA = \frac{3}{\varepsilon_0}$$

Idea behind Gauss's Law:

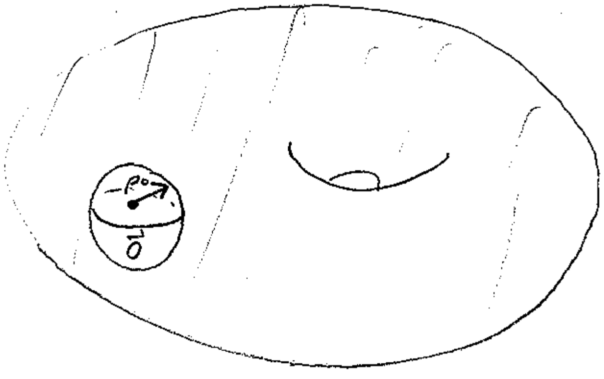
$$\iint_{\partial R} (\vec{E}\cdot\vec{n})\,dA = \iint_{\partial R} \sum (\vec{E}_i\cdot\vec{n})\,dA = \sum \iint_{\partial R} (E_i\cdot\vec{n})\,dA$$

So we can just consider the situation when we have a single charge. Let's use coordinates where that charge is at $\vec{0}$, so that $E(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{\rho^3}\vec{r}$ as before.

Key: $\displaystyle\iint_{\partial R} (\vec{E}\cdot\vec{n})\,dA = \begin{cases} 0 & \text{if } \vec{0} \text{ is not in } R \\ Q/\varepsilon_0 & \text{if } \vec{0} \text{ is in } R. \end{cases}$

Point: If $\vec{0}$ is not in R, then $\operatorname{div}\vec{E} = 0$ on all of R, so $\displaystyle\iint_{\partial R} (\vec{E}\cdot\vec{n})\,dA = \iiint_R \operatorname{div}\vec{E}\,dV = 0.$

If $\vec{o}$ is in R, consider $R' = R$ with a ball of radius ρ_0 about $\vec{o}$ removed



Then

$$0 = \iiint_{R'} \operatorname{div} E \, dV$$

$$= \iint_{\partial R'} E \cdot \vec{n} \, dA = \iint_{\partial R} E \cdot \vec{n} \, dA + \underbrace{\iint_{S_{\rho_0}} (E \cdot \vec{n}) \, dA}_{\text{with \underline{inward normal}.}}$$

Thus

$$\iint_{\partial R} (E \cdot \vec{n}) \, dA = \underbrace{\iint_{S_{\rho_0}} (E \cdot \vec{n}) \, dA}_{\text{outward normal}} = Q/\varepsilon_0 .$$

If time remains, discuss charge density and Maxwell's First Eqn.