Lecture 55: Final Review:

Reminder: Office Hours, etc on Final Handout.

Differential Forms: Things can be integrated over manifolds of the right dimension: they do this by eating vectors.

[Point: provide a unified approach to integration theorems, which we rather add hoc, because of 3-dimensionality.]

In $\mathbb{R}^2$ and $\mathbb{R}^3$ everything we do with forms can also be done with vectorfields, curl, div, etc. Not true in higher dims, [cross prod et special to dim 3/4.] and in e.g. special relativity it is best to think of $\vec{E} = (E_1, E_2, E_3)$, $\vec{B} = (B_1, B_2, B_3)$

$$F = E_1\, dx\wedge dt + E_2\, dy\wedge dt + E_3\, dz\wedge dt + B_1\, dy\wedge dz + B_2\, dz\wedge dx + B_3\, dx\wedge dy \quad \text{on } \mathbb{R}^4$$

write up ahead of time

$$J = \rho\, dt + J_1\, dx + J_2\, dy + J_3\, dz$$ (J_1, J_2, J_3) current density

(otherwise, figuring out behavior under Lorentz transformations is tricky.)

Maxwell: $dF = 0$ and $d(*F) = 4\pi(*J)$ ← see review problems.

Clearly invariant under L. transforms.

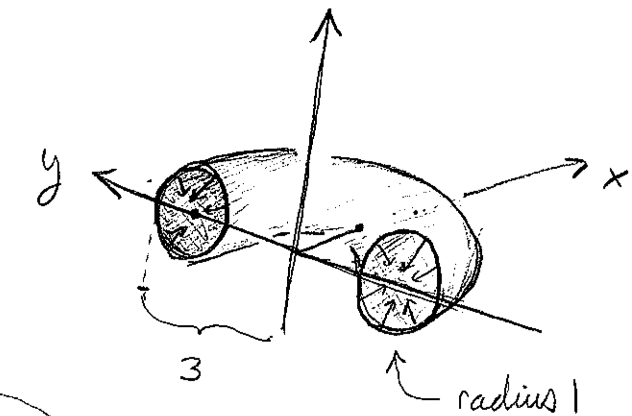
[Even though diff forms on $\mathbb{R}^2$ and $\mathbb{R}^3$ are equivalent to things about vector field, curl, etc, you may find it easiest to just forget this — the relationship is somewhat convoluted.]

Main Things: 1) How to integrate forms.

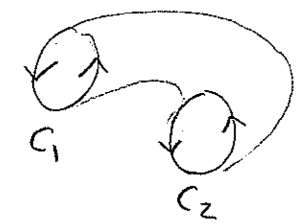
2) the d operation.

3) Stokes theorem.

Ex: S is



with inward normal



Check Stokes' theorem for

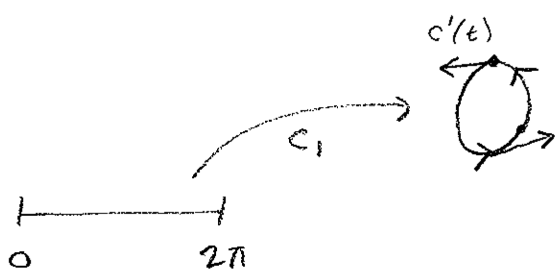
$$\alpha = (x^2+y)\,dz$$

$$\int_{\partial S}\alpha = \int_{C_1}\alpha + \int_{C_2}\alpha$$ and we can param these curves as

$$c_1(t) = (0, 2+\cos t, -\sin t)$$
$$c_2(t) = (0, -2+\cos t, -\sin t) \qquad 0 \le t \le 2\pi$$

$$\int_{C_1}\alpha = \int_0^{2\pi}\alpha_{c_1(t)}(c_1'(t)) = \int_0^{2\pi}(\cos t + 2)\,dz(c_1'(t)) = \int_0^{2\pi} -(\cos t+2)\cos t\,dt$$

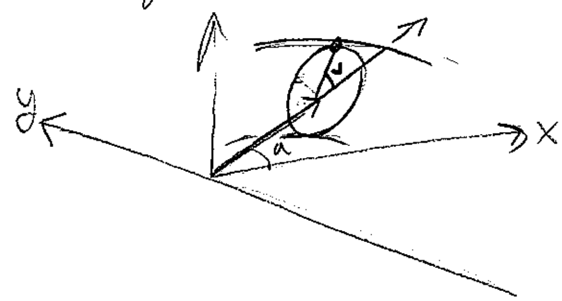
$$= \int_0^{2\pi} -\cos^2 t + 2\cos t\,dt = -\pi$$



Similarly, $\int_{C_2}\alpha = -\pi$ and

hence $\int_{\partial S}\alpha = -2\pi$

Now we need to parameterize the surface



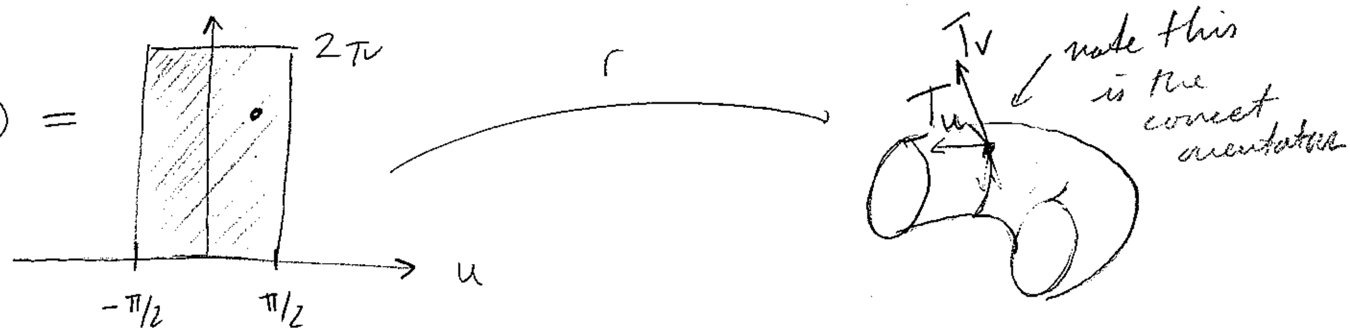
$r(u,v) =$

$(2\cos u, 2\sin u, 0) +$

$(\cos v \cos u, \cos v \sin u, -\sin v)$

$= ((2+\cos v)\cos u, (2+\cos v)\sin u, -\sin v)$

on $D =$



$d\alpha = d(x^2+y)\wedge dz = (2x\,dx + dy)\wedge dz = 2x\,dx\wedge dz + dy\wedge dz$

and so

$$\int_S d\alpha = \iint_D d\alpha_{r(u,v)}(T_u, T_v) = \int_{\pi/2}^{\pi/2}\int_0^{2\pi} 2(2+\cos v)\, dx\wedge dz(T_u,T_v) + dy\wedge dz(T_u,T_v)\; dv\,du$$

$$= \int_{\pi/2}^{\pi/2}\int_0^{2\pi} 2(2+\cos v)\begin{vmatrix} dx(T_u) & dx(T_v) \\ dz(T_u) & dz(T_v)\end{vmatrix} + \begin{vmatrix} dy(T_u) & dy(T_v) \\ dz(T_u) & dz(T_v)\end{vmatrix}\; dv\,du$$

As $T_u = (-(2+\cos v)\sin u, (2+\cos v)\cos u, 0)$

$T_v = (-\sin v \cos u, -\sin v \sin u, -\cos v)$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} 2(2+\cos v) \begin{vmatrix} -(2+\cos v)\sin u & -\sin v \cos u \\ 0 & -\cos v \end{vmatrix} + \begin{vmatrix} (2+\cos v)\cos u & -\sin v \sin u \\ 0 & -\cos v \end{vmatrix} \, du\, dv$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} + 2(2+\cos v)^2 \sin u \cos v - (2+\cos v)\cos v \cos u \; dv\, du$$

$$= \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \text{same } du\, dv = \int_0^{2\pi} 2(2\cos v - \cos^2 v) = -2\pi$$

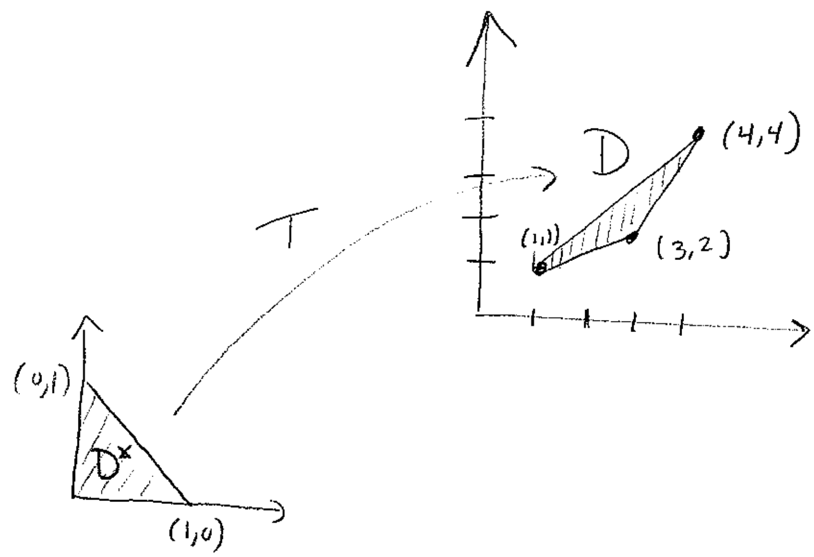
Thus:

$$\int_{\partial S} \alpha = -2\pi = \int_S d\alpha$$

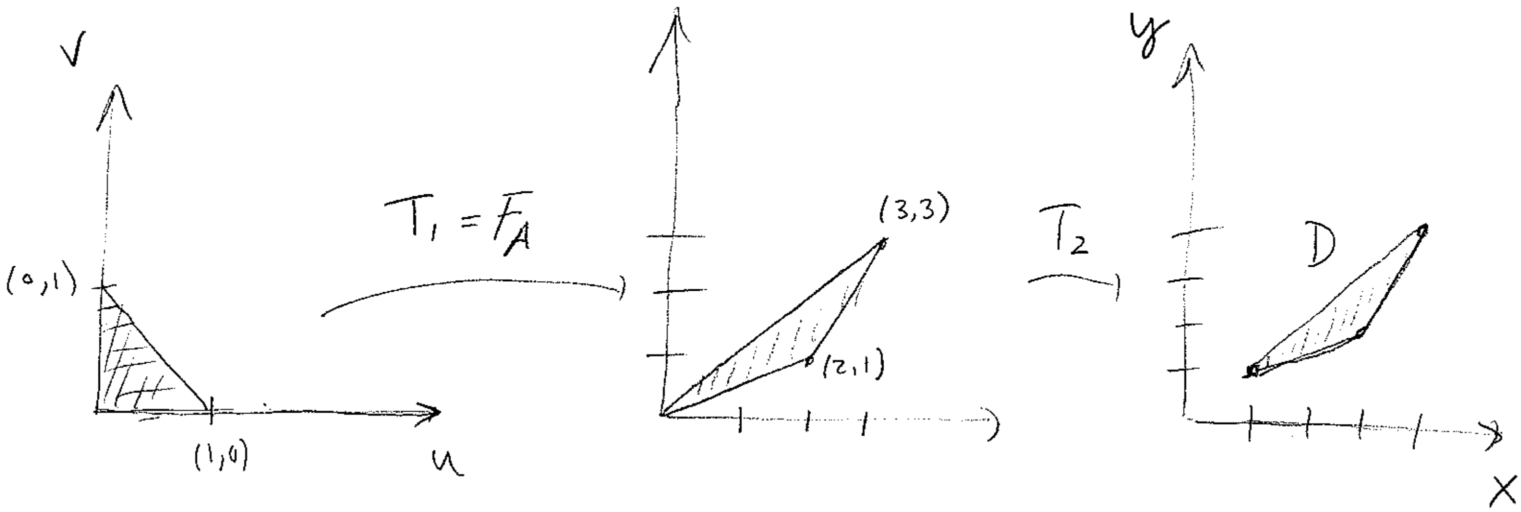
as per Stokes' Theorem!

Change of Variables:

$$\iint_D (x^2+y^2)\, dA$$



Favorite Kind of T: linear, but that won't work here



$$A = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} \qquad T_2(x,y) = (x+1, y+1)$$

$$T_1(u,v) = (2u+3v, u+3v)$$

$$T(u,v) = T_2(T_1(u,v)) = (2u+3v+1, u+3v+1)$$

$$\iint_D x^2+y^2\, dA = \int_0^1\int_0^{1-v} ((2u+3v+1)^2 + (u+3v+1)^2)\, \underbrace{|\det DT(u,v)|}_{=3}\, du\, dv.$$

Comments on final:

- Comprehensive.
- Problems somewhat harder than previous exams, but not 3 times as many.
- Same basic format.
- Any questions?