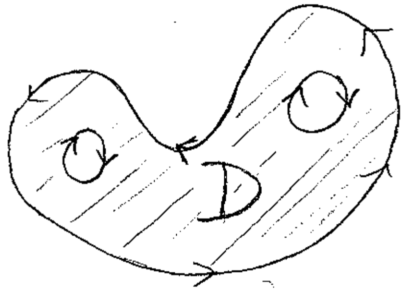
Lecture 41: The Divergence Theorem in $\mathbb{R}^2$

HW: Handout

Next time: Gauss's Law.

Last time:

Green's Thm:

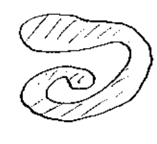


← oriented so region is to the left.

$\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ vector field

$$\int_{\partial D} \vec{F} \cdot ds = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

In practice one side is easier to evaluate than the others. E.g. finding the area of a strange region in parametric form



Divergence: $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ a vector field $\vec{F} = (F_1, F_2)$

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

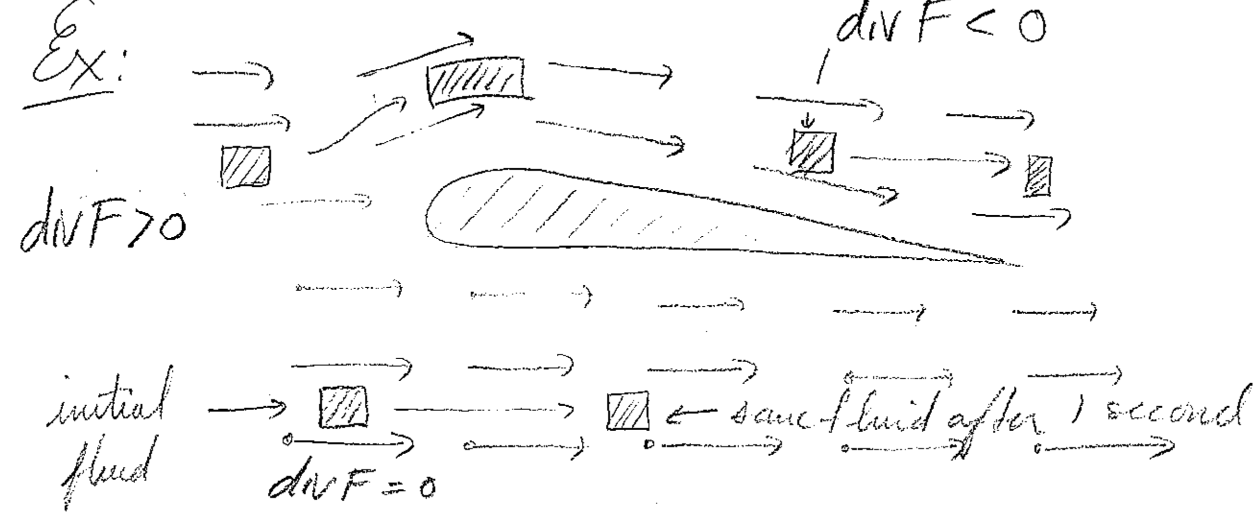
which is [Q.] a function $\mathbb{R}^2 \to \mathbb{R}$

[See Section 4.6]

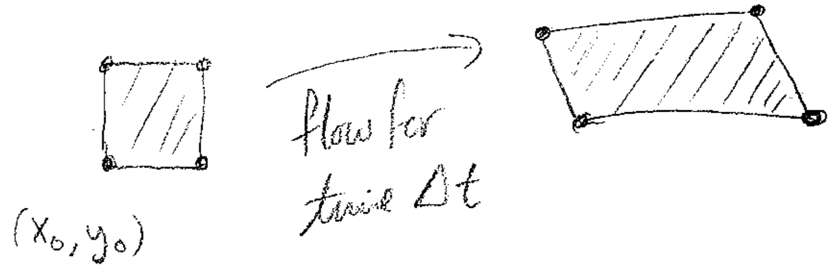
Meaning: [If $\vec{F}$ represents fluid flow, then]

$\operatorname{div} \vec{F}$ = rate of expansion under the flow

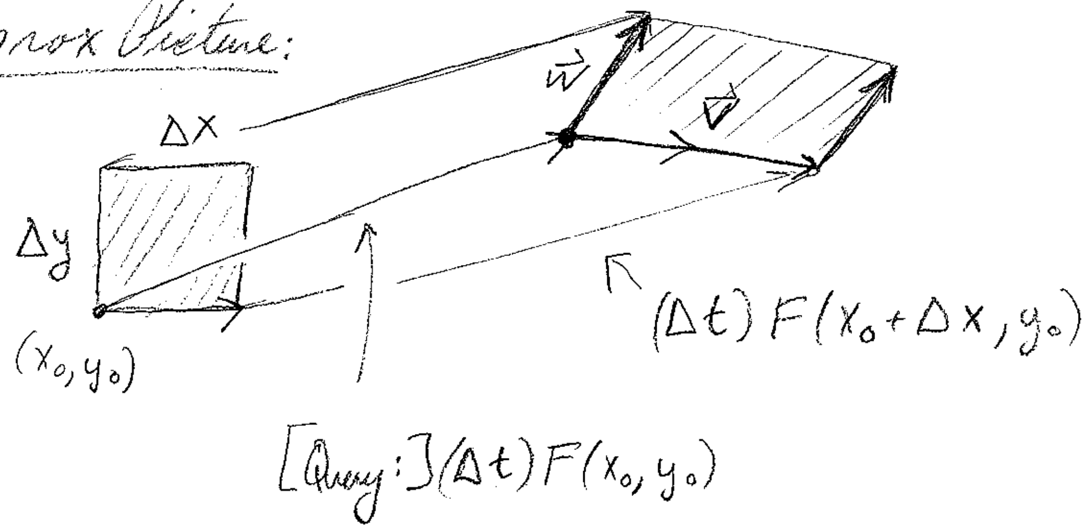
Ex:



Why?



Approx Picture:



$$\vec{v} \approx (\Delta x, 0) + \Delta t \Delta x \frac{\partial F}{\partial x}(x_0, y_0) = \Delta x \left(1 + \Delta t \frac{\partial F_1}{\partial x}, \Delta t \frac{\partial F_2}{\partial x}\right)$$

$$\vec{w} \approx \Delta y \left(\Delta t \frac{\partial F_1}{\partial y}, 1 + \Delta t \frac{\partial F_2}{\partial y}\right)$$

So the new area is

$$= \Delta x \Delta y \left(1 + \Delta t \left(\operatorname{div} \vec{F}(x_0, y_0)\right) + \Delta t^2 (\text{stuff})\right)$$

Thus:

$$\frac{\text{new area}}{\text{old area}} \approx 1 + \Delta t (\operatorname{div} \vec{F})$$

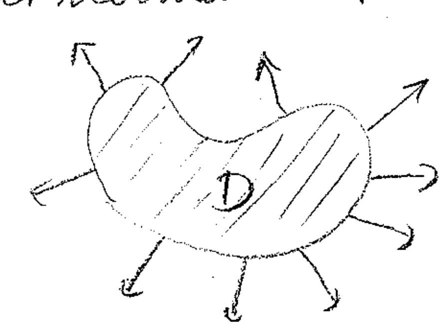
and so the rate of expansion is $(\operatorname{div} \vec{F})$.

Check units: $\vec{F}(x,y)$ has units m/s.

$\operatorname{div} \vec{F}$ has units 1/s, and so $1+\Delta t \operatorname{div} \vec{F}$ is dimensionless.

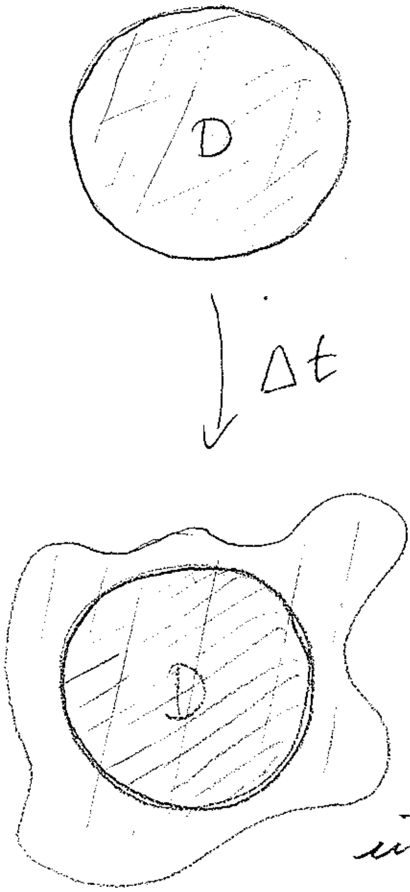
Divergence Theorem: D region in $\mathbb{R}^2$, ∂D has outward unit normals $\vec{n}$.

$$\underbrace{\int_{\partial D} \vec{F}\cdot\vec{n}\, ds}_{\text{rate fluid is crossing } \partial D} = \iint_D \operatorname{div} F\, dA$$



Reason: Consider the fluid in D which has area A. After time Δt, this fluid has area $\approx A + \Delta t\, r$ where

$$r = \iint_D \operatorname{div} F\, dA$$



Now the area of D is fixed, so the change in area can only be accomplished by fluid crossing ∂D. The amount of fluid crossing ∂D in time Δt is about $\Delta t \int_{\partial F} \vec{F}\cdot\vec{n}\, ds$

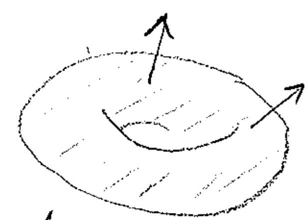
Thus $\int_{\partial D} F \cdot ds = \iint_D \operatorname{div} F \, dx\, dy$.

Connection to Green's Theorem (see HW).

If $F: \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field, we can also talk about the divergence.

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

<u>Divergence Theorem</u>: D a region in $\mathbb{R}^3$.



$\vec{F}$ a vector field. $\vec{n}$ outward normals for ∂D.

$$\iint_{\partial D} \vec{F} \cdot n \, dA = \iiint_D \operatorname{div} \vec{F} \, dV$$

This can be interpreted in the same way as before.