# Volume 54: Cohomology to Cosmology

## Closed and Exact Forms: [Recall conservative vector fields]

An $n$-form $\alpha$ is closed if $d\alpha = 0$,
exact if $\alpha = d\beta$.

Note: 1) Exact forms are closed as $d\alpha = d(d\beta) = 0$

2) If $\alpha = F_1\,dx + F_2\,dy + F_3\,dz \longleftrightarrow \vec{F} = (F_1, F_2, F_3)$

Thus

function

$\alpha$ is exact, i.e. $\alpha = df \iff \vec{F} = \text{grad } f$

$\alpha$ closed $\iff \text{curl } \vec{F} = \vec{0}$

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Reminders: • Pick up final handout
• Send.

<u>Thm</u>: Suppose $W$ is a star-shaped region in $\mathbb{R}^n$. If $\alpha$ is a closed form, it is exact.

For more complicated spaces, this isn't true and leads to <u>cohomology</u>, an important topological invariant

$$\underset{\text{manifold}}{H^n(M)} = \left\{ \begin{array}{l} \text{All closed } n\text{-forms where} \\ \text{we declare } \alpha = \beta \text{ if } \alpha - \beta \text{ is exact} \end{array} \right\}$$

<u>Ex</u>: 1) $M$ star shaped $H^n(M) = 0$ for all $n$

2) $M =$ 

non-zero elt is answer from 10(b).

$$H^2(M) = H^0(M) = \mathbb{R}$$

constant functions.

$$H^n(M) = 0 \text{ for all others.}$$

3)

$M =$ 

$$H^2(M) = H^0(M) = \mathbb{R}$$
$$H^1(M) = \mathbb{R}^{2n}$$

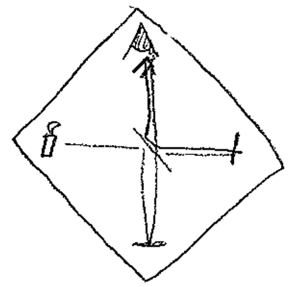
$n$ - holes

Can be used to prove the Brouwer Fixed Point Thm.

# What 3-manifold is the universe?

A brief history of physics:

Late 19th century: Maxwell's equations have solutions which look like waves moving at speed c. Suggests these propagate in some medium "the ether", [Actually, "ether" was a motivation for Maxwell...] which is needed to provide the "absolute frame of reference". In 1887, Michelson-Morley are unable to detect Earth's motion through the ether.



1905: Special Relativity: time is not absolute, but the speed of light is!

Use $\mathbb{R}^4$ but there is no define time variable, depends on the particular observer.

1916: General Relativity: matter curves space

1917: Einstein applies theory to universe as a whole

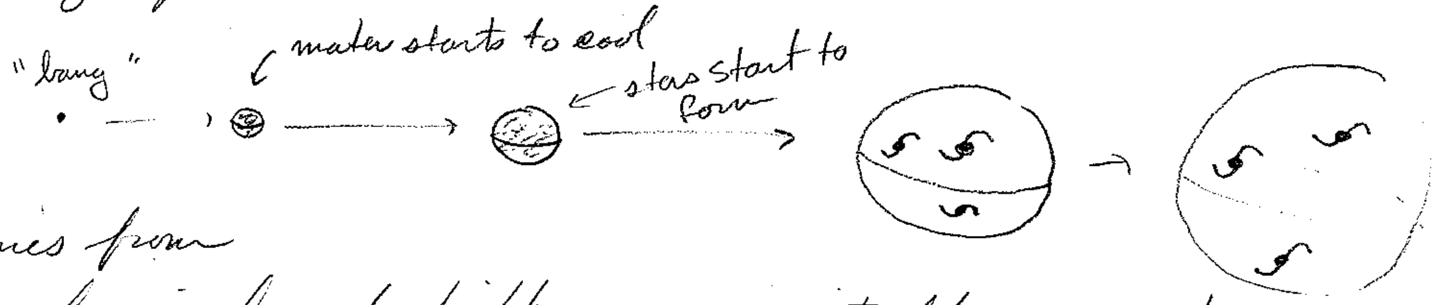
Chooses "spacelike slice" to be $S^3$, not $\mathbb{R}^3$

Finds universe collaspes, which he dislikes and so intoduces "the cosmological constant" (empty space has mass) to correct for this.

1927-9: Lemaître/Hubble discovers that the universe is expanding at a rate of $\approx 7\%$/billion years!

leads to "Big Bang" picture of the universe.

10-15 billion years ago, all space/matter was compressed to single point.

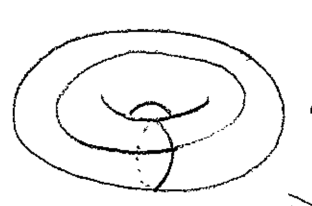


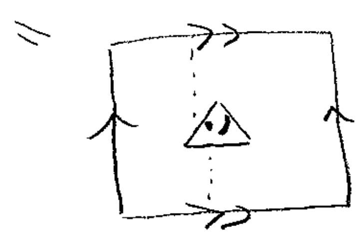

Comes from
cosmological red shift — space itself is expanding

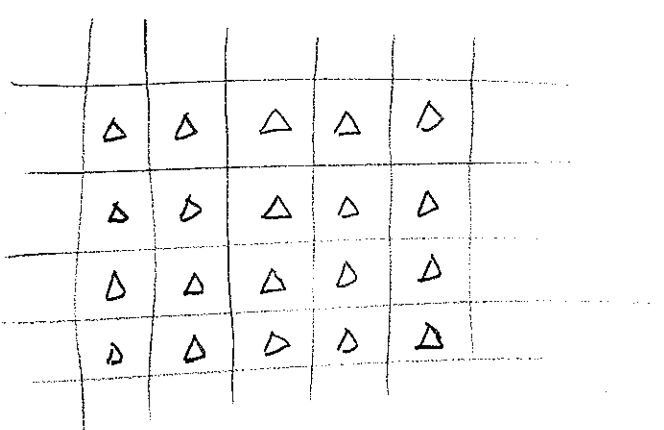
[Hubble noticed greater red shifts at greater distances, which can't be explained by the doppler effect w/o making us the (literal) center of the universe]

Is the universe $\mathbb{R}^3$ or something compact like $S^3$?

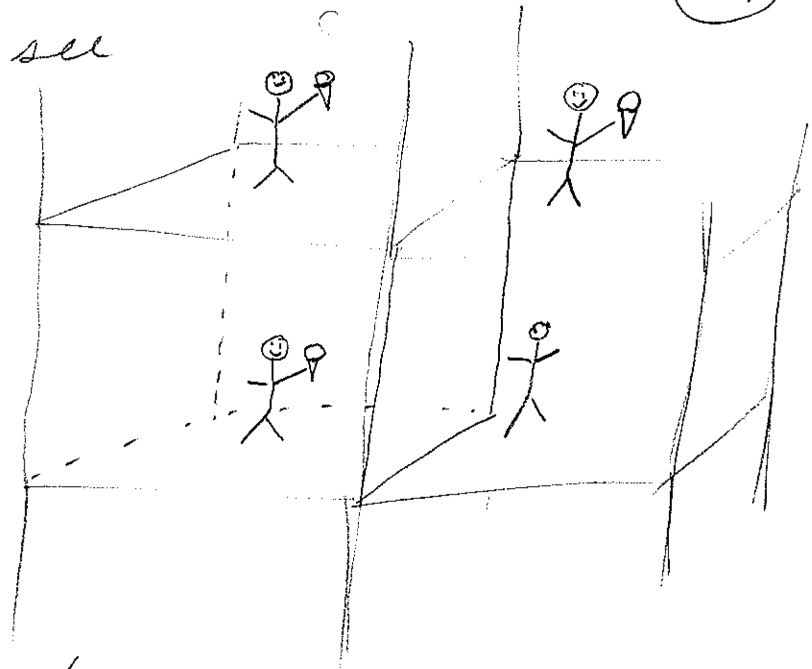
How could we tell?

Suppose we live in . What do we see?





If we live in $T^3$, we'd see



So if the universe is compact, we could look out in the night sky and see our own Milky Way!

[Not that we could tell, and it would be a copy back in time...]

How we might be able to find out:

Match circles in the CMB, etc, etc.

<u>See</u>: Jeff Weeks, "The shape of space".

http://geometrygames.org