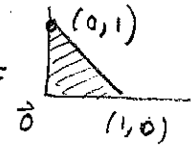
Lecture 52: Why Stokes Theorem works.

Last time:

Stokes Theorem: M an n-manifold, α an $(n-1)$-form on M. Then $\int_{\partial M} \alpha = \int_M d\alpha$.

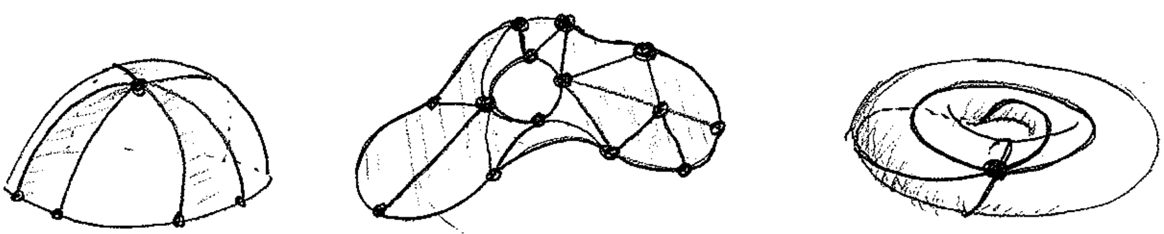
[Review heuristic and physical justifications given for our variants of this. But now's the time to give a mathematical justification...]

Focus on surfaces ($n=2$) just so we can visualize things easily. Turns out, the key case is

(4) Stokes For $\triangle$s: Let $T =$ 

and α a 1-form on T. Then $\int_{\partial T} \alpha = \int_T d\alpha$.

Consider a general surface S. We can break it into $\triangle$s (with curved sides), call them $T_1, \ldots, T_K$



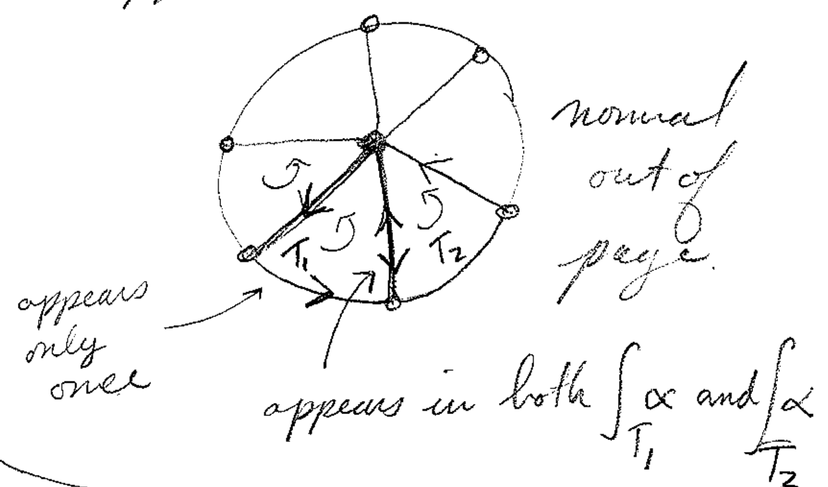
[Here edges are match up to other edges.]

Now if α is a 1-form on S, suppose the Stokes theorem holds for each T_i. Then

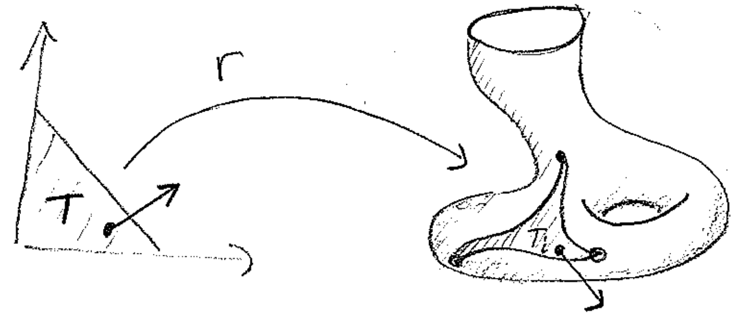
$$\int_S d\alpha = \sum_{i=1}^{k} \int_{T_i} d\alpha = \sum_{i=1}^{k} \int_{\partial T_i} \alpha = \int_S \alpha$$

By assumption

Since interior edges all appear twice with opposite orientations:



Now, the presumption that $\int_{T_i} d\alpha = \int_{\partial T_i} \alpha$ actually follows from ⊛ by choosing a param $r : T \to T_i$



Then $\displaystyle\int_{\partial T_i} \alpha = \int_{\partial T} r^*(\alpha)$ where $r^*(\alpha)(\vec{v}) = \alpha_{r(u,v)}(Dr(\vec{v}))$

In particular
$$r^*(\alpha)(1,0) = \alpha(T_u)$$
etc.

Also define $r^*(d\alpha)$ by $r^*(d\alpha)(\vec{v},\vec{w}) = d\alpha(Dr(\vec{v}), Dr(\vec{w}))$

More concretely, $r^*(d\alpha) = d\alpha(T_u, T_v)\, du \wedge dv$

So

omitted calculation

$$\int_{\partial T_i} \alpha = \int_{\partial T} r^*(\alpha) \overset{\text{by } (\star)}{=} \int_T d(r^*(\alpha)) = \int_T r^*(d\alpha)$$

$$= \int_{T_i} d\alpha$$

as desired.

Proof of $(\star)$ Write $\alpha = \underbrace{F_1\, dx}_{\alpha_1} + \underbrace{F_2\, dy}_{\alpha_2} = \alpha_1 + \alpha_2$

If we know $\int_{\partial T} \alpha_i = \int_T d\alpha_i$ then we'll done since

$$\int_{\partial T} \alpha = \int_{\partial T} \alpha_1 + \int_{\partial T} \alpha_2 \quad \text{and} \quad \int_T d\alpha = \int_T d\alpha_1 + \int_T d\alpha_2 .$$

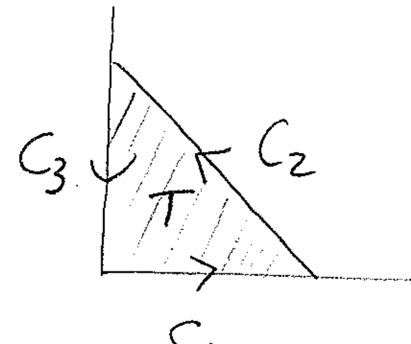
Lets do $\alpha_2 = F_2\, dy$. Then $d\alpha_2 = \frac{\partial F_2}{\partial x}\, dx \wedge dy$ and

so

$$\int_T d\alpha_2 = \int_0^1 \int_0^{1-y} \frac{\partial F_2}{\partial x}\, dx\, dy = \int_0^1 F_2(x,y)\Big|_{x=0}^{x=1-y} dy$$

$$= \int_0^1 F(1-y, y)\,dy - \int_0^1 F(0,y)\,dy + 0$$

$$= \int_{C_2} \alpha_2 + \int_{C_3} \alpha_2 + \int_{C_1} \alpha_2 = \int_{\partial T} \alpha_2$$



as needed. The case of α_1 is similar.

Closed and Exact Forms: [Recall conservative vector fields.]

An n-form α is closed if $d\alpha = 0$

it is exact if $\alpha = d\beta$.

Note: 1) Exact forms are closed, since $d\alpha = d(d\beta) = 0$.

2) If $\alpha = F_1\,dx + F_2\,dy + F_3\,dz$, consider $\vec{F} = (F_1, F_2, F_3)$

Then

α is exact, i.e. $\alpha = df$ (function) $\iff \vec{F} = \text{grad } f$

α is closed $\iff \text{curl } \vec{F} = \vec{0}$ everywhere.