Lecture 52: General Stokes Theorem

HW: Handout

Last time:

D a region in $\mathbb{R}^2$

$\alpha = F_1\, dx + F_2\, dy$ a 1-form

$$d\alpha = \left(\frac{\partial F_1}{\partial x} dx + \frac{\partial F_1}{\partial y} dy\right) \wedge dx + \left(\frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy\right) \wedge dy$$

$$= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx \wedge dy$$

Green's Theorem:

$$\boxed{\int_{\partial D} \alpha = \int_D d\alpha}$$

Since

$$\int_{\partial D} \alpha = \int_{\partial D} \vec{F} \cdot ds \quad \text{where } F = (F_1, F_2)$$

$\Vert \leftarrow$ By old Green's Theorem

and $\displaystyle\int_D d\alpha = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dx\, dy.$

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To explain the other cases, we first need to understand the connection between integrating 2-forms and vector fields over surfaces.

$$\vec{F} = (F_1, F_2, F_3) \longleftrightarrow \underbrace{F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy}_{\alpha}$$

note somewhat odd order of terms.

Suppose

$$r : D \longrightarrow S$$

is a parameterization of a surface in $\mathbb{R}^3$.

<u>Point</u>: $(dy \wedge dz) \wedge dx = (dz \wedge dx) \wedge dy = (dx \wedge dy) \wedge dz = dx \wedge dy \wedge dz$

Then by a calculation, we find

$$\vec{F}(r(u,v)) \cdot (T_u \times T_v) = \alpha_{r(u,v)}(T_u, T_v)$$

just take
$T_u = (a_1, a_2, a_3)$
$T_v = (b_1, b_2, b_3)$
and expand.

and hence

$$\iint_S (\vec{F} \cdot \vec{n})\, dA = \int_S \alpha$$

<u>Important note:</u> When integrating forms, you need to make sure your parameterizations respect your choice of a normal vector.

Similarly, you can check that if

$$\vec{F} = (F_1, F_2, F_3) \text{ is a vector field}$$

and

$$\alpha = F_1\,dx + F_2\,dy + F_3\,dz$$

then

$$\text{curl } \vec{F} \longleftrightarrow d\alpha$$

under the correspondence between vector fields and 2-forms.

So Stokes Theorem for a surface S in $\mathbb{R}^3$ and vector field $\vec{F}$ becomes

$$\begin{array}{ccc} \int_{\partial S} \vec{F} \cdot ds & = & \iint_S (\text{curl } \vec{F}) \cdot \vec{n}\, dA \\ \| & & \| \\ \int_{\partial S} \alpha & & \int_S d\alpha \end{array}$$

That is: $\int_{\partial S} \alpha = \int d\alpha$ which is exactly the same statement as Green's Theorem!

What about the Divergence Theorem?

W region in $\mathbb{R}^3$.

α a 2-form on W. Then

$$\int_{\partial W} \alpha = \int_W d\alpha$$

since if we write

$$\alpha = F_1 \, dy \wedge dz + F_2 \, dz \wedge dx + F_3 \, dx \wedge dy$$

then

$$d\alpha = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx \wedge dy \wedge dz$$

so

$$\int_{\partial W} \alpha = \int_{\partial W} (\vec{F} \cdot \vec{n}) \, dA \quad \text{where } \vec{F} = (F_1, F_2, F_3)$$

$\Vert \leftarrow$ equal by Divergence Theorem.

$$\int_W d\alpha = \iiint_W \operatorname{div} \vec{F} \, dA$$

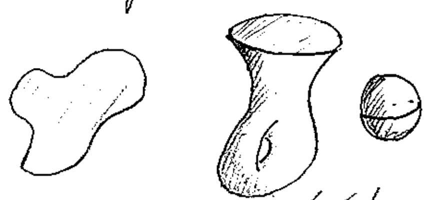
General Stokes Theorem: M an n-manifold. α an $(n-1)$-form on M. Then $\displaystyle\int_{\partial M} \alpha = \int_M d\alpha$

What's a manifold? Some examples:

Curves

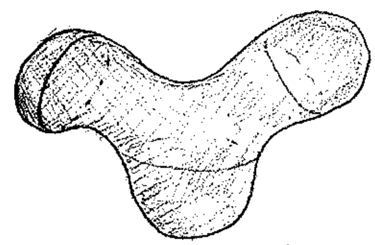
1-manifold

Surfaces



2-manifolds

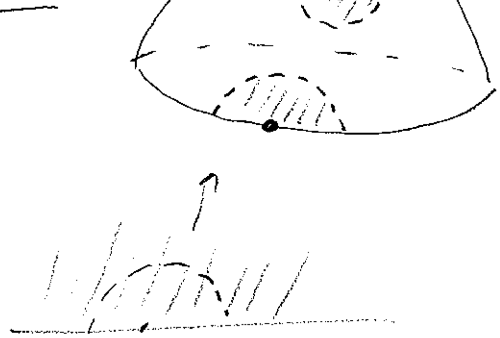
Regions in $\mathbb{R}^3$



3-manifolds

Def: An n-manifold is something which locally looks like $\mathbb{R}^n$, except at boundary points where it looks like $\{\vec{x} \text{ in } \mathbb{R}^n \text{ with } x_1 \geq 0\}$.

Ex:



I study 3-manifolds often without boundary, e.g. $S^3, T^3, \ldots$

[Talk about this if time permits.]