Lecture 51: More on differential forms

HW: Section 8.4 #1, 3, 5, 7, 9, 13, 23, 26, 28, 29, 34, 37

The story so far on $\mathbb{R}^3$:

1-forms: $y^2\,dx + z\,dy + z^2\,dz$

$$\underbrace{\int_C}_{\text{curve}} \alpha = \int_a^b \alpha_{c(t)}(c'(t))\,dt$$

c a param. of C.

2-forms: $y\,dx\wedge dy + xz\,dy\wedge dz$

$$\underbrace{\int_S}_{\text{surface}} \alpha = \iint_D \alpha_{r(u,v)}(T_u, T_v)\,du\,dv$$

$r: D \to S$ a param.

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3-forms: $f(x,y,z)\,dx\wedge dy\wedge dz$

If W is a region in $\mathbb{R}^3$, α a 3-form, then

$$\int_W \alpha = \iiint f(x,y,z)\,dx\,dy\,dz$$

There are no non-zero 4-forms on $\mathbb{R}^3$ [or 5-forms, ...] though there are on $\mathbb{R}^4$ [or $\mathbb{R}^5$...]

Notes: 1) None of these integrals depend on the choice of parameterization

2) For 1-forms α, β, γ,

$$\underbrace{(\alpha\wedge\beta\wedge\gamma)_p}_{\text{3-form}}(v_1, v_2, v_3) \stackrel{!}{=} \underbrace{\begin{vmatrix} \alpha_p(v_1) & \alpha_p(v_2) & \alpha_p(v_3) \\ \beta_p(v_1) & \beta_p(v_2) & \beta_p(v_3) \\ \gamma_p(v_1) & \gamma_p(v_2) & \gamma_p(v_3) \end{vmatrix}}_{\text{measures "volume"}}$$

Wedge Product: k-form α, j-form $\beta \rightsquigarrow \alpha \wedge \beta$ a $(k+j)$ form

[Have already seen this for 1-forms]

Rules: Associativity: $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$

Anticommutativity: $\beta \wedge \alpha = (-1)^{kj} \alpha \wedge \beta$

Ex: 1) $dy \wedge dx = -dx \wedge dy$ 2) $dx \wedge dx = -dx \wedge dx = 0$

3) $(dy \wedge dz) \wedge dx = dy \wedge dz \wedge dx = -dy \wedge dx \wedge dz$
$= dx \wedge dy \wedge dz = dx \wedge (dy \wedge dz)$

Distributivity:

$$(\alpha + \beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma$$

Example:

$$(y\, dy \wedge dz + x\, dx \wedge dz) \wedge (x\, dx + dy)$$

can pull functions out in front

$= (y\, dy \wedge dz) \wedge (x\, dx) + (\ldots)$ $= -(dy \wedge dy) \wedge dz = 0$

$= xy\, dy \wedge dz \wedge dx + y\, dy \wedge dz \wedge dy + 0 + x\, dx \wedge dz \wedge dy$

$= (xy - x)\, dx \wedge dy \wedge dz$

Differentiating Forms: k-form $\alpha \rightsquigarrow (k+1)$-form $d\alpha$

1) For a 0-form on $\mathbb{R}^3$, i.e. $f: \mathbb{R}^3 \to \mathbb{R}$

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$$

2) $d(\alpha+\beta) = d\alpha + d\beta$ and $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$

where α is a k-form.

3) $d(d\alpha) = 0$

Ex: 1) For the zero-form $f(x,y,z) = x$,

$$\underset{\displaystyle d(x)}{\overset{\shortparallel}{df}} = 1\,dx + 0\,dy + 0\,dz = dx$$

, so at least the notation is consistient.

2) $\alpha = y\,dx + (x^2+z^2)\,dz$

$$d\alpha = d(y) \wedge dx + y\,d(dx) + d\big((x^2+z^2)\,dz\big)$$

$$= dy \wedge dx + d(x^2+z^2)\,dz = -dx \wedge dy + (2x\,dx + 2z\,dz) \wedge dz$$

$$= -dx \wedge dy + 2x\,dx \wedge dz$$

Check:

$$d(d\alpha) = 0 + d(2x) \wedge dx \wedge dz = 0 + 2\,dx \wedge dx \wedge dz = 0.$$

Properly interpreted, the "d" operation on differential forms encompasses div, grad, and curl and be used to reinterpret our various integral theorems.

Consider $\vec{F} = (F_1, F_2)$ a vector field on a region D in $\mathbb{R}^2$

<u>Green's Theorem:</u>

$$\int_{\partial D} \vec{F} \cdot ds = \iint_D \frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial y} \, dx\,dy$$

Let $\alpha = F_1\,dx + F_2\,dy$; then $\int_{\partial D} \alpha = \int_{\partial D} \vec{F} \cdot ds$. In addition,

$$d\alpha = dF_1 \wedge dx + dF_2 \wedge dy = \left(\frac{\partial F_1}{\partial x}\,dx + \frac{\partial F_1}{\partial y}\,dy\right) \wedge dx + dF_2 \wedge dy$$

$$= \frac{\partial F_1}{\partial y}\,dy \wedge dx + \frac{\partial F_2}{\partial x}\,dx \wedge dy = \left(\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial x}\right) dx \wedge dy$$

So $\int_D d\alpha = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$, Thus,

<u>Green's Theorem, Mark II:</u> D a region in $\mathbb{R}^2$;
α a 1-form.

Then $$\int_{\partial D} \alpha = \int_D d\alpha .$$