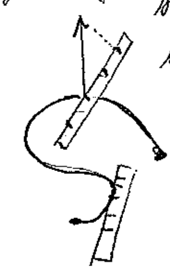
Lecture 50: 2-forms and integration.

HW: Section 8.4 # 23, 28, 29

Evaluations: Textbook / ICES online

So far we've seen the following objects on $\mathbb{R}^n$:

Was used to measure vectors and so do integrals over curves.



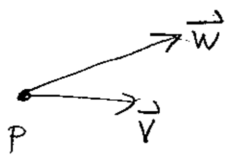
functions (aka 0-forms)	point in $\mathbb{R}^n$ $\longmapsto$	a number
Vector fields	point in $\mathbb{R}^n$ $\longmapsto$	a vector
1-form	point in $\mathbb{R}^n$ $\longmapsto$	(linear transformation $\mathbb{R}^n \to \mathbb{R}$ ("ruler"))
2-form	point in $\mathbb{R}^n$ $\longrightarrow$	("Antisymmetric bilinear form" on $\mathbb{R}^n$)

something that given two vectors gives the "area" of the parallelogram they span.

Some examples of 2-forms:

If α and β are 1-forms, then there is a 2-form $\alpha \wedge \beta$ given by:

$$(\alpha\wedge\beta)_p(\vec{v},\vec{w}) = \begin{vmatrix} \alpha_p(\vec{v}) & \alpha_p(\vec{w}) \\ \beta_p(\vec{v}) & \beta_p(\vec{w}) \end{vmatrix} = \alpha_p(\vec{v})\beta_p(\vec{w}) - \alpha_p(\vec{w})\beta_p(\vec{v})$$



Notice: 1) $(\alpha\wedge\beta)_p(\vec{w},\vec{v}) = -(\alpha\wedge\beta)_p(\vec{v},\vec{w})$

2) $(\alpha\wedge\beta)_p(\vec{v}_1+\vec{v}_2,\vec{w}) = (\alpha\wedge\beta)_p(\vec{v}_1,\vec{w}) + (\alpha\wedge\beta)_p(\vec{v}_2,\vec{w})$

3) $(\beta\wedge\alpha)_p(\vec{v},\vec{w}) = -(\alpha\wedge\beta)_p(\vec{v},\vec{w})$, that is $\beta\wedge\alpha = -\alpha\wedge\beta$.

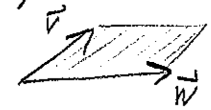
Ex: On $\mathbb{R}^2$ have basic 1-forms dx, dy where for $\vec{v} = (v_1, v_2)$

$$dx_p(\vec{v}) = v_1 \quad \text{and} \quad dy_p(\vec{v}) = v_2$$

So

$$dx \wedge dy\,(\vec{v}, \vec{w}) = \begin{vmatrix} dx(\vec{v}) & dx(\vec{w}) \\ dy(\vec{v}) & dy(\vec{w}) \end{vmatrix} = \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} =$$

signed area of the parallelogram spanned by



Any 2-form on $\mathbb{R}^2$ can be written

$$f(x,y)\, dx \wedge dy.$$

On $\mathbb{R}^3$, we have basic 1-forms dx, dy, dz and 2-forms can be written

$$\alpha = f(x,y,z)\, dx \wedge dy + g(x,y,z)\, dx \wedge dz + h(x,y,z)\, dy \wedge dz$$

Aside: Book doesn't always write the wedges.

Ex $\quad \alpha = z^2 dx \wedge dy + (x+y)\, dy \wedge dz$

$$P = (1,0,2) \quad \vec{v} = (1,2,3) \quad \vec{w} = (0,1,0)$$

$$\alpha_p(\vec{v}, \vec{w}) = 4\, dx \wedge dy(\vec{v}, \vec{w}) + dy \wedge dz(\vec{v}, \vec{w})$$

$$= 4 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} = 1$$

Integrating 2-forms:

For a region D in $\mathbb{R}^2$ and 2-form α,

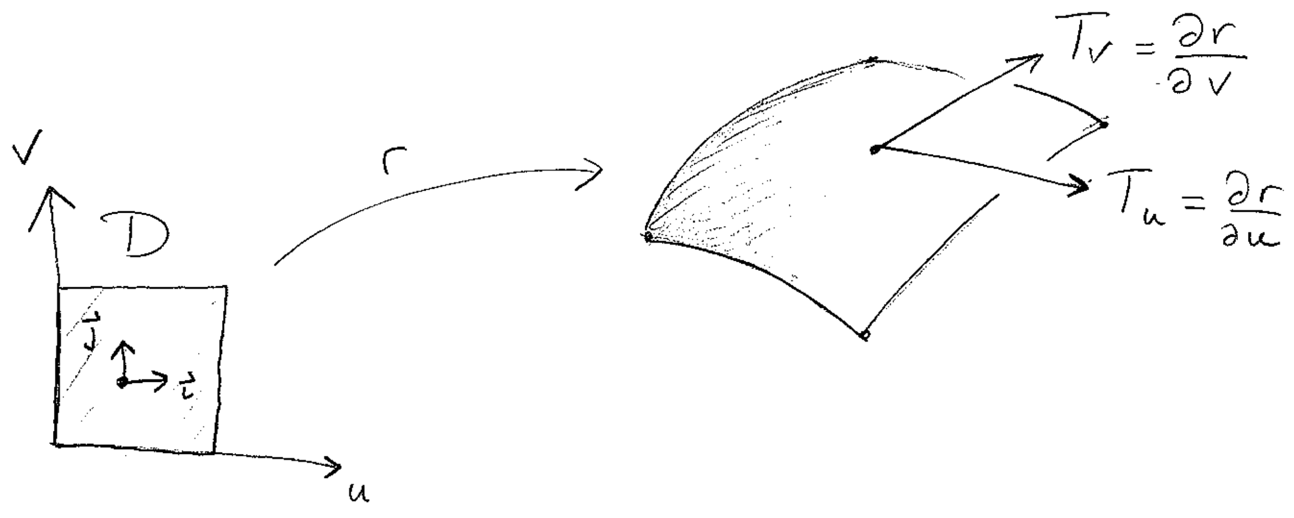
$$\int_D \alpha = \iint_D f(x,y)\,dx\,dy \quad \text{where} \quad \alpha = f(x,y)\,dx \wedge dy$$

traditional not to write $\iint$ here

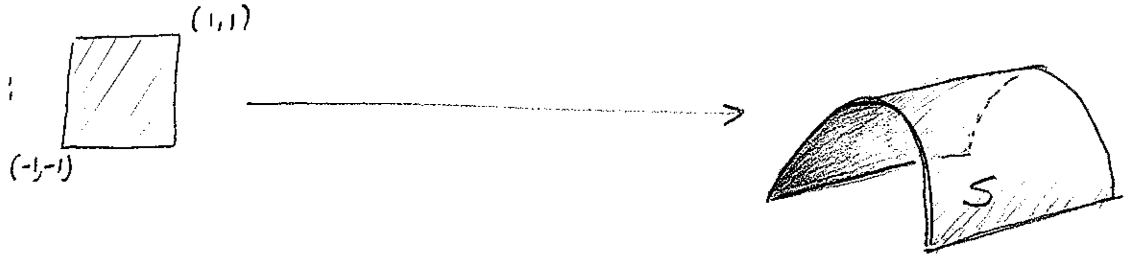
For a surface S in $\mathbb{R}^3$ and 2-form α,

$$\int_S \alpha = \iint_D \alpha_{r(u,v)}(T_u, T_v)\,du\,dv$$

where $r : D \to S$ is our parametrization



Ex: r:



$$r(u,v) = (u, v, 1-u^2)$$

$$\alpha = z^2\, dx \wedge dy + (x+y)\, dy \wedge dz$$

$$\int_S \alpha = \iint_D \alpha_{r(u,v)}(T_u, T_v)\, du\, dv$$

Now $T_u = (1, 0, -2u)$ and $T_v = (0, 1, 0)$, so

$$\alpha_{r(u,v)}(T_u, T_v) = (1-u^2)^2\, dx \wedge dy\, (T_u, T_v) + (u+v)\, dy \wedge dz\, (T_u, T_v)$$

$$= (1-u^2)^2 + (u+v) \cdot (2u)$$

since

$$dx \wedge dy\, (T_u, T_v) = \begin{vmatrix} dx(T_u) & dx(T_v) \\ dy(T_u) & dy(T_v) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$dy \wedge dz\, (T_u, T_v) = \begin{vmatrix} dy(T_u) & dy(T_v) \\ dz(T_u) & dz(T_v) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -2u & 0 \end{vmatrix} = 2u$$

So

$$\int_S \alpha = \int_{-1}^{1} \int_{-1}^{1} (1-u^2)^2 + (u+v)(2u)\, du\, dv = \frac{24}{5}$$