Lecture 49: Differential Forms (§8.4)

This week: §8.4   Note: ICES online.

So far, we've a bunch of theorems relating integrals over boundaries with integrals of derivatives over regions.
[div, grad, curl...]
very 3-dim'l.
These are all the same if we use the language of differential forms:

> ω a k-form defined on a region R of dimension $k+1$.
> Then:
> $$\int_{\partial R} \omega = \int_R d\omega$$

The goal of this week is to explain what this means.
[Applications: "repackaging" Maxwell's equations; fixing the Sleipner A.]

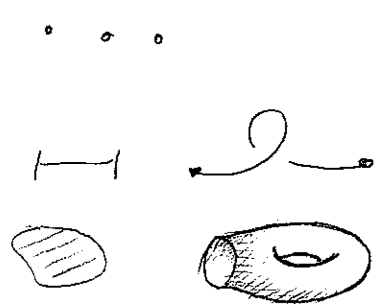
A k-form is something which can be integrated over a k-dimensional object. ["The bit after the integral sign."]

Ex: 0-form: $f(x)$

1-form: $f(x)\,dx$ $f(x,y,z)\,ds$

2-form: $f(x,y)\,dx\,dy$ dA

[3-form: $f(x,y,z)\,dx\,dy\,dz$ dV]



0-forms: A 0-form on U in $\mathbb{R}^n$ is just a function: $f: U \to \mathbb{R}$

1-form: A 1-form on U in $\mathbb{R}^3$ is something of the form

$$\alpha = f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz$$

where $f, g, h: U \to \mathbb{R}$ are functions.

for the moment these are formal symbols, [playing a role like $\vec{i}, \vec{j}, \vec{k}$ for defining vector fields.]

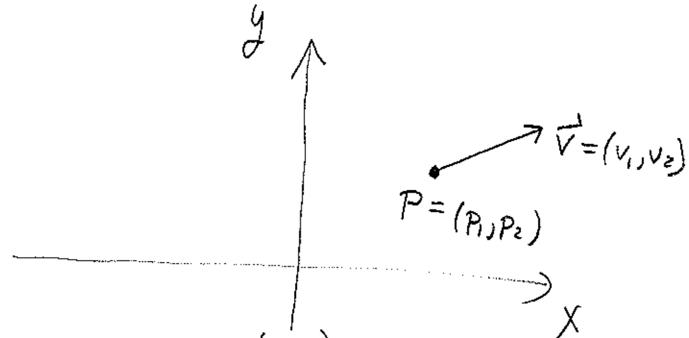
Ex: $\alpha = y^2\,dx + (x+y)\,dy$ a 1-form on $\mathbb{R}^2$

[Remind how this looks like something you can integrate over a path.]

What a 1-form does: Eats vectors

If $\alpha = f(x,y)\,dx + g(x,y)\,dy$

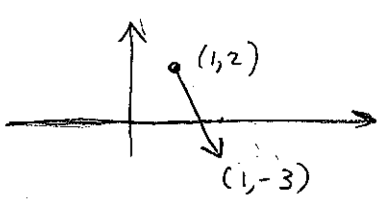
then



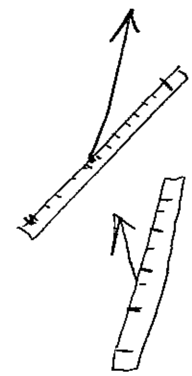
$$\alpha_p(\vec{V}) = f(p)V_1 + g(p)V_2 = \begin{pmatrix} f(p) & g(p) \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

Ex: α as above

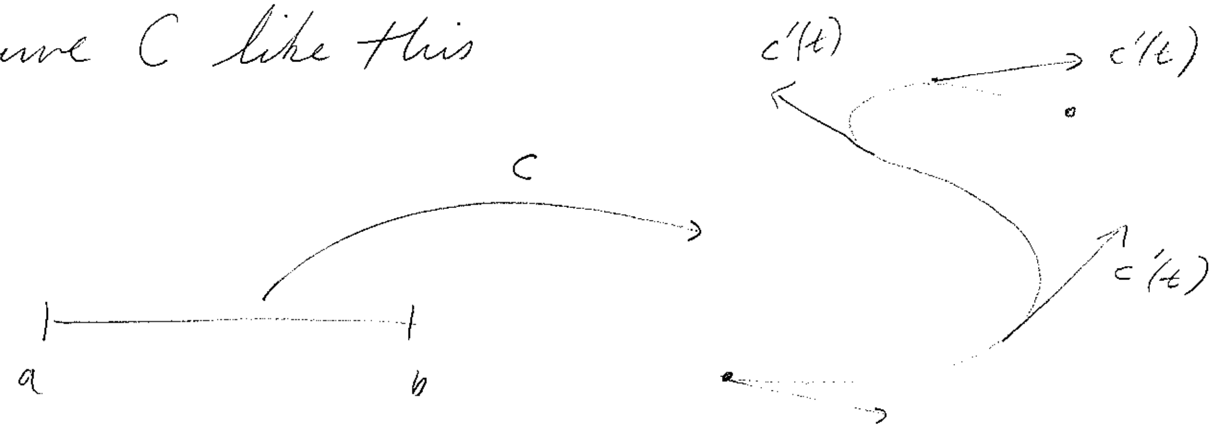
$$\alpha_{(1,2)}(1,-3) = \begin{pmatrix} 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = 4 - 9 = -5$$



Thus α assigns to each point in the plane a linear transformation $\mathbb{R}^2 \to \mathbb{R}$. We can think of this as a "ruler" at each point which we use to measure vectors.



We can integrate a 1-form over a a curve C like this



$$\int_C \alpha = \int_a^b \alpha_{c(t)}(c'(t))\, dt$$

$$= \int_a^b f(c(t))c_1'(t) + g(c(t))c_2'(t)\, dt$$

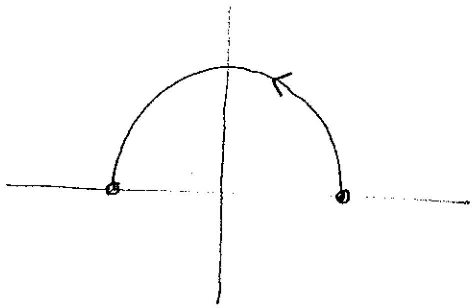
where

$$\alpha = f(x,y)\,dx + g(x,y)\,dy$$

and

$$c(t) = (c_1(t), c_2(t)).$$

As always, this doesn't depend on the parameterization.

<u>Ex</u>: $C =$ 

$$c(t) = (\cos t, \sin t)$$
$$0 \le t \le \pi$$

$$\alpha = y^2\,dx + (x+y)\,dy$$

$$\int \alpha = \int_0^{\pi} \alpha_{c(t)}(c'(t))\,dt = \int_0^{\pi} (\sin^2 t\ ,\ \cos t + \sin t)\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt$$

$$= \int_0^{\pi} -\sin^3 t + \cos^2 \pi + \sin t \cos t \; dt =$$

<u>What's going on here?</u>

1) Before, we needed two things to integrate over a curve: a function and "ds".
The 1-form α combines these 2 into one package.

2) A 1-form can be thought of as coming from a vector field $\vec{F}$ where

$$\alpha_p(\vec{v}) = \vec{F}(p) \cdot \vec{v} \quad \text{Then} \int_C \alpha = \int_C \vec{F} \cdot ds.$$